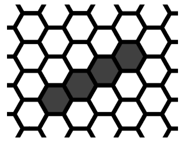
International Mathematical Olympiad
«Formula of Unity» / «The Third Millennium»
Year 2022/2023. Final round

Problems for grade R5

Each problem is worth 7 points. Criteria for individual tasks are printed in grey.

1. Tom and Jerry play on an infinite field of hexagonal cells. Initially, only 4 cells are black, and all other cells are white (see the picture). In a move, a player recolors one cell: Tom makes a black cell white, and Jerry makes a white cell black. Tom starts the game. It is forbidden to recolor the cell that the opponent has just colored. Tom wins when there are no two adjacent black cells on the field (i. e. all black cells are separated from each other). Can Jerry continue the game indefinitely and why? (*O. Pyayve*)



Solution. After Tom's first move, in any case, two black adjacent cells will remain (let's call them cells A and B). It is enough for Jerry to make a "triangle" of three cells (that is, to color one of the two hexagons adjacent to both A and B – let's call it C). If Tom repaints one of the cells A, B, C (e.g., A), the other two will remain adjacent, that is, Jerry will not lose. Moreover, Jerry will have the opportunity to "restore" the triangle by coloring another hexagon adjacent to B and C on the other side. (And if Tom colored another cell, Jerry can make a random move.) Thus, Jerry can act so that the game continues indefinitely.

Criteria. Correct answer, but incorrect or absent Jerry's strategy — 0 points.

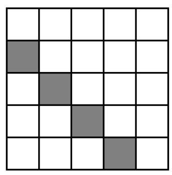
2. There are 49 equal squares. Make two rectangles out of them so that their perimeters are 2 times different. All squares should be used. (*L. Koreshkova*)

Answer: 7×4 and 1×21 or 1×33 and 1×16.

Solution. The sum of their areas is indeed 49 ($28 + 21$ in the first case and $33 + 16$ in the second one), and their perimeters indeed differ by 2 times: $2 \cdot (7 + 4) = 22$, $2 \cdot (1 + 21) = 44$ or $2 \cdot (1 + 16) = 34$, $2 \cdot (1 + 33) = 68$.

Criteria. The correct answer is given without calculations showing that it fits — 5 points.
At least one of the sides is not an integer — 0 points.

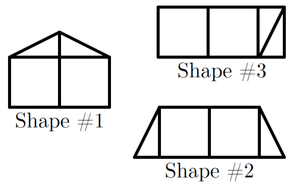
3. The cells of the square 5×5 contain natural numbers from 1 to 5 so that in each column, in each row and in each of the two main diagonals all numbers are different. Can the sum of the numbers in the gray cells (see the picture) be 20? (*L. Koreshkova*)



Solution. Let it be possible. Then, in order for the sum of the numbers in the gray cells to be 20, all four numbers in them must be «5». Then it remains to write only one number «5» in the square, while both diagonals still do not contain this number, so it must be placed in the center of the board, but then there will be two numbers «5» in the central row and column. A contradiction.

Criteria. The idea is given that all gray cells contain «5» — 3 points.

4. Irene has two identical squares and two identical triangles. She made of them three shapes shown in the picture, and then measured the perimeters of these shapes. She obtained 26 for the first shape, 32 for the second and 30 for the third one. Find the lengths of the sides of the triangle. (*L. Koreshkova*)



Answer: 3, 4, 5.

Solution. Let's denote the sides of the triangle: we'll call the shortest one s, the middle one — m, the longest one — l. The perimeter of the figure #2 differs from the perimeter of the figure #1 by two short sides of the triangle, so $s = (32 - 26) : 2 = 3$. Now we can get the length of the middle side of the triangle from the figure #3 — indeed, its perimeter consists of two short sides and six middle ones, so $m = (30 - 2s) : 6 = 4$. Finally, now from the figure #1 we will find the longest side: $l = (26 - 4m) : 2 = 5$.

Criteria. Incorrect answer (except arithmetic error) and/or partial solution, leading nowhere — 0 points.

Correct answer without any explanation — 2 points. Correct answer with only checking and/or lengths guessing — 3 points.

5. On Pi Day (March 14), the participants of the spring math camp decided to give circles to all their friends, and squares to those they just know. Andrew noticed that each boy had received 3 circles and 8 squares, and each girl 2 squares and 9 circles. And Kate calculated that in total 4046 figures were donated. Prove that one of them is wrong. (*P. Mulenko*)

Solution. Each participant of the camp was given exactly $3 + 8 = 2 + 9 = 11$ figures. Then the total number of figures is equal to the product of 11 by the number of participants, but 4046 is not divisible by 11.

Criteria. The idea of an equal number of figures among participants — 3 points.

Not leading anywhere ideas about the number of boys and/or girls — 0 points.

6. How many numbers from 1 to 999 **without zeros** are written in Roman numerals exactly one character longer than in decimal notation? (*P. Mulenko*)

Note. To write a number in Roman numerals, you need to break it into decimal summands, write each summand in accordance with the table, and then write them down sequentially from largest to smallest. For example, for the number 899, we have $800 = \text{DCCC}$, $90 = \text{XC}$, $9 = \text{IX}$, so we get DCCCXCIX.

1 I	10 X	100 C	1000 M
2 II	20 XX	200 CC	2000 MM
3 III	30 XXX	300 CCC	3000 MMM
4 IV	40 XL	400 CD	
5 V	50 L	500 D	
6 VI	60 LX	600 DC	
7 VII	70 LXX	700 DCC	
8 VIII	80 LXXX	800 DCCC	
9 IX	90 XC	900 CM	

Answer: 68.

Solution. Regardless of the digit position and other digits of the number, the decimal digit a is written as:

- one symbol for $a = 1$ and $a = 5$,
- two symbols for a equal to 2, 4, 6, 9,
- three symbols for $a = 3$ and $a = 7$,

- four symbols at $a = 8$.

This means that only the digits 1, 5 and exactly one of the digits 2, 4, 6, 9 are used in suitable numbers (otherwise the total length of the record will be longer by more than one character). That is, 4 single-digit numbers are suitable, $4 \cdot 2 + 2 \cdot 4 = 16$ two-digit numbers and $4 \cdot 2 \cdot 2 + 2 \cdot 4 \cdot 2 + 2 \cdot 2 \cdot 4 = 48$ three-digit numbers. 68 numbers in total.

Criteria. Correct answer without justification — 2 points.

One of the cases described above is lost or extra cases are accounted for — 0 points.

The condition about the absence of zero is incorrectly interpreted (for example, as «excluding zero», that is, that the length of the Roman notation of a number should be one more than the number of non-zero digits — no more than 3 points.

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Problems for grade R6

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1. Eugene and Paul have a regular hexagon. They play a game: in one move, a player places a positive integer in any free vertex. After six moves, when the game ends, a referee writes on each side the product of the numbers in its ends. Then all 12 numbers are added up. If the sum is odd, then Eugene wins, and otherwise Paul.
   It is known that Eugene starts the game. Who can win regardless of the actions of the opponent and how should he act? (*L. Koreshkova*)

   Solution. Let the vertices be numbered from 1 to 6 clockwise and divided into 3 pairs of diametrically opposite ones (1–4, 2–5, 3–6).

   In order for Paul to win, he should put the same number as Eugene put the previous move into the second vertex in the pair, so the sum of the numbers in each pair of vertices will be even. In this case, all the edges will also split into three pairs with the same numbers on the ends (1–2 and 4–5, 2–3 and 5–6, 3–4 and 6–1), therefore, the sum of the numbers in the edges will also be even, which means that the total result will be even as well.

   Criteria. Correct answer without any explanation — 0 points.

   If symmetry appears in the incorrect strategy for Paul or the symmetric strategy is incorrectly justified — 3 points.

2. There are 81 equal squares. Make two rectangles out of them so that their perimeters are equal. All squares should be used. (*L. Koreshkova*)

   Answer: 3×11 and 6×8.

   Solution. The sum of their areas is $33 + 48 = 81$, and their perimeters are indeed equal: $2 \cdot (3 + 11) = 2 \cdot (6 + 8) = 28$.
   Remark This is the only correct answer (in other words, there are no other matching pairs of rectangles).

   Criteria. The correct answer is given without calculations showing that it fits — 5 points.

   At least one of the sides is not an integer — 0 points.

3. Eight boys (Adam, Ben, Charlie, Daniel, Harry, Jack, Lucas and Oliver) stood one after another in some order, so that they have numbers from 1 to 8 in the line. They noticed that:

   - Ben's number is three times larger than Daniel's number;
   - Oliver stands somewhere after the third boy, but before Harry;
   - Adam's number is half of Jack's number;
   - the fourth boy stays just after Lucas and somewhere before Jack.

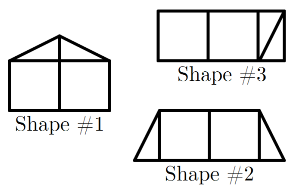
   What order were the boys in? Explain why you think so. (*P. Mulenko*)

   Answer: Charlie, Daniel, Lucas, Adam, Oliver, Ben, Harry, Jack.

Solution. According to the fourth condition, Lucas is in third place. Since Ben's number is divided by 3 according to the first condition, and the third number is occupied, then Ben has the number 6, and Daniel — the number 2. From the third condition it follows that the Jack's number is even and is not equal to 4 (since Daniel has already taken the number 2), so Jack has the number 8, and Adam — 4. Then the numbers of Oliver and Harry are obtained from the second condition (5 and 7, respectfully), and Charlie gets the remaining number 1.

Criteria. Correct answer without explanation or with only verification of the conditions — 1 point.
Correct explanations of the placement of pairs of guys (Ben and Daniel, Adam and Jack, Oliver and Harry) — +2 points for each pair.

4. Irene has two identical squares and two identical triangles. She made of them three shapes shown in the picture, and then measured the perimeters of these shapes. She obtained 74 for the first shape, 84 for the second and 82 for the third one. Find the lengths of the sides of the triangle. (*L. Koreshkova*)



Answer: 5, 12, 13.

Solution. Let's denote the sides of the triangle: we'll call the shortest one s, the middle one — m, the longest one — l. The perimeter of the figure #2 differs from the perimeter of the figure #1 by two short sides of the triangle, so $s = (84 - 74) : 2 = 5$. Now we can get the length of the middle side of the triangle from the figure #3 — indeed, its perimeter consists of two short sides and six middle ones, so $m = (82 - 2s) : 6 = 12$. Finally, now from the figure #1 we will find the longest side: $l = (74 - 4m) : 2 = 13$.

Criteria. Incorrect answer (except arithmetic error) and/or partial solution, leading nowhere — 0 points.
Correct answer without any explanation — 2 points. Correct answer with only verification and/or lengths guessing — 3 points.

5. From 50 to 70 children came to the spring math camp. On Pi Day (March 14), they decided to give circles to all their friends, and squares to those they just know. Andrew calculated that each boy had received 3 circles and 8 squares, and each girl 2 squares and 9 circles. And Kate found that the same number of circles and squares were donated in total. How many children came to the camp? (*P. Mulenko*)

Solution. Let's denote the number of boys as b, and girls as g. Then $3b + 9g = 8b + 2g$. So, $5b = 7g$, thus the numbers of boys and girls correlate as $7 : 5$. So, the total number of children is divisible by 12. Between 50 and 70, only the number 60 is suitable.

Criteria. A misunderstood condition (for example, that total amounts of figures given to boys and girls are equal) — 0 points.
Correct answer without explanation — 2 points. Correct answer with only verification — 3 points.

6. How many numbers from 1 to 999 **without zeros** are written in Roman numerals with the same number of characters as in decimal notation? (*P. Mulenko*)

Note. To write a number in Roman numerals, you need to break it into decimal summands, write each summand in accordance with the table, and then write them down sequentially from largest to smallest. For example, for the number 899, we have 800 = DCCC, 90 = XC, 9 = IX, so we get DCCCXCIX.

1 I	10 X	100 C	1000 M
2 II	20 XX	200 CC	2000 MM
3 III	30 XXX	300 CCC	3000 MMM
4 IV	40 XL	400 CD	
5 V	50 L	500 D	
6 VI	60 LX	600 DC	
7 VII	70 LXX	700 DCC	
8 VIII	80 LXXX	800 DCCC	
9 IX	90 XC	900 CM	

Answer: 52.

Solution. Regardless of the digit position and other digits of the number, the decimal digit a is written as:

- zero symbols for $a = 0$,
- one symbol for $a = 1$ and $a = 5$,
- two symbols for a equal to 2, 4, 6, 9,
- three symbols for $a = 3$ and $a = 7$,
- four symbols at $a = 8$.

Thus, in order for a number to meet the condition, it must:

1) either consist **only** of digits 1 and 5,
2) or contain of **a pair** of digits 0 and x, where x is one of the digits 2, 4, 6 or 9 and, zero or one digit 1 or 5,
3) or consist of one digit 3 or 7 and two zeros.

There are $2 + 2 \cdot 2 + 2 \cdot 2 \cdot 2 = 14$ numbers of the first type and 2 numbers of the third type (DCC and CCC). There are exactly 4 two-digit numbers of the second type, and $2 \cdot 2 \cdot 4 \cdot 2 = 32$ three-digit ones (2 options to put a digit «0», the 2 options to put a single-symbol digit, then 4 options to choose one of two-symbol digits and, at last, 2 to options to choose a single-symbol digit). $14 + 2 + 4 + 32 = 52$ numbers in total.

Criteria. Correct answer without justification — 2 points.

-2 points for any lost case of the described above.

Arithmetic error in any of the cases —-1 point for each one.

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Problems for grade R7

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1. 20 people met in a math circle. Among them, there were exactly 49 pairs of people who knew each other before. Prove that someone knew at most 4 participants. (*L. Koreshkova*)

   Solution. Assume the opposite. Let everyone know at least 5 people. Then, before the meeting, there were at least $20 \cdot 5/2 =$50 pairs of familiar people. A contradiction.

   Criteria. The problem is solved under the assumption that everyone knew exactly 5 people — 5 points.

2. There are 81 equal squares. Make two rectangles out of them so that their perimeters are 2 times different. All squares should be used. (*L. Koreshkova*)

   Answer: 3×19 and 3×8 or 3×23 and 1×12.

   Solution. The sum of their areas is indeed 81 ($57 + 24$ in the first case and $69 + 12$ in the second one), and their perimeters indeed differ by 2 times: $2 \cdot (3 + 8) = 22$, $2 \cdot (3 + 19) = 44$ or $2 \cdot (1 + 12) = 26$, $2 \cdot (3 + 23) = 52$.

   Criteria. The correct answer is given without calculations showing that it fits — 5 points.
   At least one of the sides is not an integer — 0 points.

3. Two 2-digit numbers are written on the board. Andrew multiplied them and obtained a four-digit number with the first digit 2. Paul added them up and got a three-digit number. If you cross out the first digit from Andrew's number, you get Paul's number. What numbers were written? (*L. Koreshkova*)

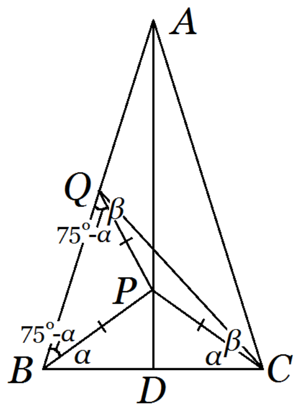
   Answer: $(24, 88)$; $(30, 70)$.

   Solution. Let's denote written numbers as x and y. Then, by the condition, $xy = 2000+x+y$, or $(x - 1)(y - 1) = 2001$. Since $2001 = 3 \cdot 23 \cdot 29$ and x, y are two-digit numbers, then $x, y \in \{24, 30, 70, 88\}$. The condition is satisfied only by the pairs $\{24, 88\}$ and $\{30, 70\}$.

   Criteria. There is an incorrect pair among the answers — 0 points.
   If it is not proved, that there are no more solutions: only one of the solutions is found — 1 point, both solutions — 2 points.
   If the solution is correct, but one of the solutions is lost during the brute force due to a logical error — 5 points.

4. Given an isosceles triangle ABC in which $\angle A = 30^\circ$, $AB = AC$. A point D is the midpoint of BC. A point P is chosen on the segment AD, and a point Q is chosen on the side AB, so that $PB = PQ$. Find the angle PQC. (*L. Koreshkova*)

Answer: 15°.



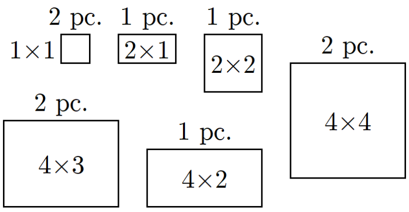
Solution. Since D is the midpoint of the base of an isosceles triangle, then AD is the median, bisector, and height of the triangle. Let's draw the segment PC. Since $\Delta PDB = \Delta PDC$ (PD is common, $DB = DC$, $\angle PDB = \angle PDC = 90^\circ$), then $PC = PB = PQ$, therefore, all three triangles ΔPBC, ΔPBQ and ΔPQC are isosceles.

Denote $\angle PBD = \angle PCD = \alpha$, from where $\angle PBQ = \angle PQB = 75^\circ - \alpha$, and $\angle PQC = \angle PCQ = \beta$. Then the sum of the angles of the triangle ΔQCB is $2\alpha + 2 \cdot (75^\circ - \alpha) + 2\beta = 180^\circ$, from where $\beta = 15^\circ$.

Criteria. Correct answer without explanation — 0 points.

It is proved that $PB = PC = PQ$ — 3 points.

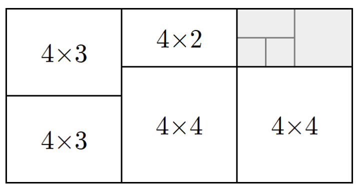
5. A few years ago, there were 9 different paintings in the computer game “Minecraft” (see the picture): two square paintings 4×4, two squares 1×1, a square 2×2, two horizontal paintings 4×3, as well as one horizontal 2×1 and one horizontal 4×2. In how many ways can all 9 paintings be placed on a rectangular wall 12 blocks long and 6 blocks high? Paintings should not overlap and cannot be rotated. (*P. Mulenko*)



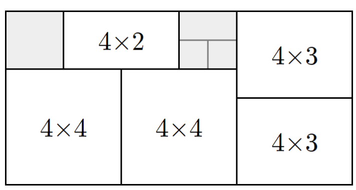
Answer: 896.

Solution. We will say that two paintings are located in different columns if no block of the first painting is in the same column with any block of the second one. It is obvious that the 4×4 paintings are in different columns with each other and with the 4×3 paintings at any placement. Thus, it is necessary for the paintings 4×3 be strictly under one another. Paintings of 4×4 will be ajacent to the horizontal edges, since there should still be place for the paintings of 4×2 and 2×2 on the wall.

Firstly, let's consider the case when 4×2 is strictly above or below one of the 4×4 pictures. It is clear that there are $3 \cdot 2^3 \cdot 2^2 = 96$ ways to arrange all the pictures with a width of 4 (if the union of small pictures is considered one picture 4×2, then we have 3 ways to choose a column for 4×3 paintings, 2^2 ways to choose floor/ceiling in each of the other columns, and 2^3 ways to swap pictures of the same size with each other). The remaining paintings can be placed in $2^3 = 8$ ways in the remaining rectangle 4×2, since the paintings 2×2 and 2×1 must be in different columns.



Secondly, let the painting 4×2 be above or below two paintings of 4×4 at once. Since it must be in different columns with the paintings 2×2 and 2×1, then there are 16 ways to arrange all the paintings with a width of 4 (2 ways to choose a column for 4×3, 2 ways to choose “ceiling/floor” and $2^2 = 4$ ways to rearrange paintings of the same size). The remaining two sections 2×2 can be filled in 8 ways.



Therefore we have $8 \cdot (96 + 16) = 896$ options in total.

Criteria. It is shown that the three smallest paintings (two 1×1 and one 1×2) can only be put together in the form of a square 2×2 — 1 point.

Correct explanation is given about the arrangement of paintings 4×4 and 4×3 — 2 more points.

Any non-arithmetic error in calculating number of arrangements of the pictures of width 4 (e.g.: it is not taken into account that the numbers of options in described cases (when the painting 4×2 is located strictly under or above single painting 4×4 and when this painting is under or above both of them) are different; or it is not taken into account that in cases where the paintings 4×4 are located in adjacent columns, the number of remaining cases varies depending on their position (both at the top or bottom and one at the top, the other at the bottom)) — -2 points.

Arithmetic error in counting cases — -1 point.

6. There are 17 islands in a kingdom far far away, and each of the islands is inhabited by 119 people. The inhabitants of the kingdom are divided into two castes: knights, who always tell the truth, and liars, who always lie. During a population census, each person was first asked: "Excluding you, is your island inhabited by equal numbers of knights and liars?". It turned out that on 7 islands everyone answered "Yes", and on the other islands, everyone answered "No". Then each person was asked: "Is it true that, including you, the people of your caste contain less than half of the population of your island?". This time, on some 7 islands, everyone answered "No", and on the other islands, everyone answered "Yes". How many liars are in the kingdom? (*P. Mulenko*)

Answer: 1013.

Solution.

1) Consider the first question. On the one hand, the answer «yes» to it will be given either by a knight on the island where exactly 60 knights are living, or by a liar on the island with another number of knights. On the other hand, the answer «no» can be given by either a liar on an island with 59 knights, or by a knight with another number of knights.
So, on those 7 islands, where everyone answered the first question «yes», there are either 60 or 0 knights; on the remaining 10 islands there are either 59 or 119 knights.
2) The second question, regardless of castes, will be answered «yes» if there are less than half of the knights on the island, and «no» otherwise. So, on those 7 islands, where everyone answered the second question «no», there are at least 60 knights (that is, 60 or 119); and on the remaining 10 islands there are no more than 59 knights (that is, 59 or 0).
3) Let there be 60 knights on x islands and 59 knights on y islands; then (see point 1) there are 0 knights on $7 - x$ islands and 119 knights on $10 - y$ islands. From point 2 we get: $x + (10 - y) = 7$, $y + (7 - x) = 10$. Both equations are equivalent to the equality $y - x = 3$.
4) The total number of knights:
$$60 \cdot x + 59 \cdot y + (7 - x) \cdot 0 + (10 - y) \cdot 119 = 60x + 59(x + 3) + 119(7 - x) = 59 \cdot 3 + 119 \cdot 7 = 1010.$$

Therefore, there are 1013 liars in the kingdom.

Criteria. All possible numbers of knights and liars on the all islands' types are correctly shown — 2 points.

The problem is solved for the case when the 7 islands in both questions are the same — 2 points.

The problem is solved correctly, but the answer indicates the number of knights, not liars — 6 points.

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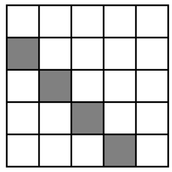
Problems for grade R8

Each problem is worth 7 points. Criteria for individual tasks are printed in grey.

1. The cells of the square 5×5 contain natural numbers from 1 to 5 so that in each column, in each row and in each of the two main diagonals all numbers are different. Can the sum of the numbers in the gray cells (see the picture) be 19? (*L. Koreshkova*)



Solution. Let it be possible. To get the sum of 19 in the gray cells, there must be three digits «5» and one digit «4» ($19 = 5 + 5 + 5 + 4$). With any arrangement of numbers «5», on the main diagonal (going from the lower left corner to the upper right) «5» can only be placed in the upper right corner, but then there isn't a place to place «5» on the side diagonal (going from the upper left corner to the lower right). A contradiction.

Criteria. It is shown that $19 = 5 + 5 + 5 + 4$ — 1 point.

If different arrangements of fives and fours in the gray cells are checked separately, and at least one of those arrangements is missing — 5 points.

2. On an island, there are $2n$ cities connected by roads so that more than n roads go out of each city. A tourist heard on the news that some two cities had to be quarantined, so all roads leading to these cities were blocked. Unfortunately he doesn't remember the names of the cities. Prove that, despite the lockdown, the tourist can still get from any opened city to any other one. (*P. Mulenko*)

Solution. Let's prove it by contradiction. Consider two unclosed cities A and B. Since each of them has at least 11 "neighbors", and there are no more than 18 neighbors in total, then at least 4 neighbors coincide. Among these coincident neighbors, at most 2 are closed, so there is an open city adjacent to both A and B. Through it, we can get from A to B.

Criteria. The problem is considered for only one example — 0 points.

The problem is solved in the case when two vertices of degree strictly $n + 1$ — 6 points.

3. Solve the equation: $[20x + 23] = 20 + 23x$. Recall that $[a]$ denotes the integer part of a number, that is, the largest integer not exceeding a. (*L. Koreshkova*)

Solution. Denote both sides of the equation by n, which is an integer. Then $x = \frac{n-20}{23}$ and $n \leqslant 20x + 23 < n + 1$. It means that

$$n \leqslant \frac{20(n-20)}{23} + 23 < n + 1,$$

that is

$$n \leqslant 43 < n + 7\frac{2}{3}.$$

It turns out that $x = 1 - \frac{k}{23}$ for an arbitrary integer $0 \leqslant k \leqslant 7$.

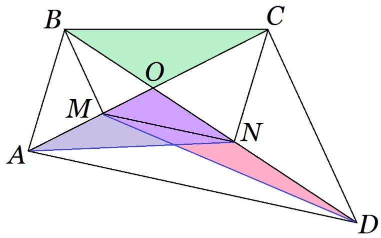
Answer: $\frac{16}{23}, \frac{17}{23}, \frac{18}{23}, \frac{19}{23}, \frac{20}{23}, \frac{21}{23}, \frac{22}{23}, 1.$

Criteria. 0 points for a solution with result $x = 1$.

2 points if it is obtained that x has the form $1 - \frac{k}{23}$.

-1 point for each missing or extra answer.

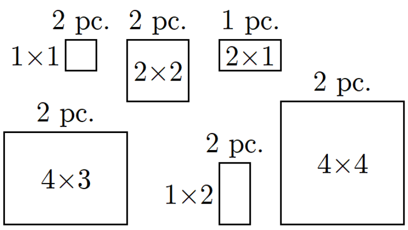
4. A quadrilateral $ABCD$ with obtuse angles B and C is given. Points M and N are chosen on the diagonals, so that $BM \parallel CD$, $CN \parallel AB$. Prove that $AD \parallel MN$. (*L. Koreshkova*)



Solution. Let O be the intersection point of the diagonals. $BCDM$ is a trapezoid ($BM \parallel DC$), so the triangles MOD and BOC have the same area. (Indeed, triangles MCD and BCD have the same area because they have a common base CD and equal heights; $S_{MOD} = S_{MDC} - S_{COD} = S_{BCD} - S_{COD} = S_{BOC}$.) Analoguosly the triangles AON and BOC have equal area.

Thus triangles MOD and AON have the same area, and hence also AMN and DMN. Since they have a common base, their heights are equal, so the points A and D are at the same distance from the line MN. Hence, $AD \parallel MN$, q. e. d.

5. A few years ago, there were 11 different paintings in the computer game "Minecraft" (see the picture): two square paintings 4×4, two 2×2, two 1×1, two horizontal paintings 4×3, as well as one horizontal 2×1 and two vertical 1×2. In how many ways can all 11 paintings be placed on a rectangular wall 12 blocks long and 6 blocks high? Paintings should not overlap and cannot be rotated. (*P. Mulenko*)



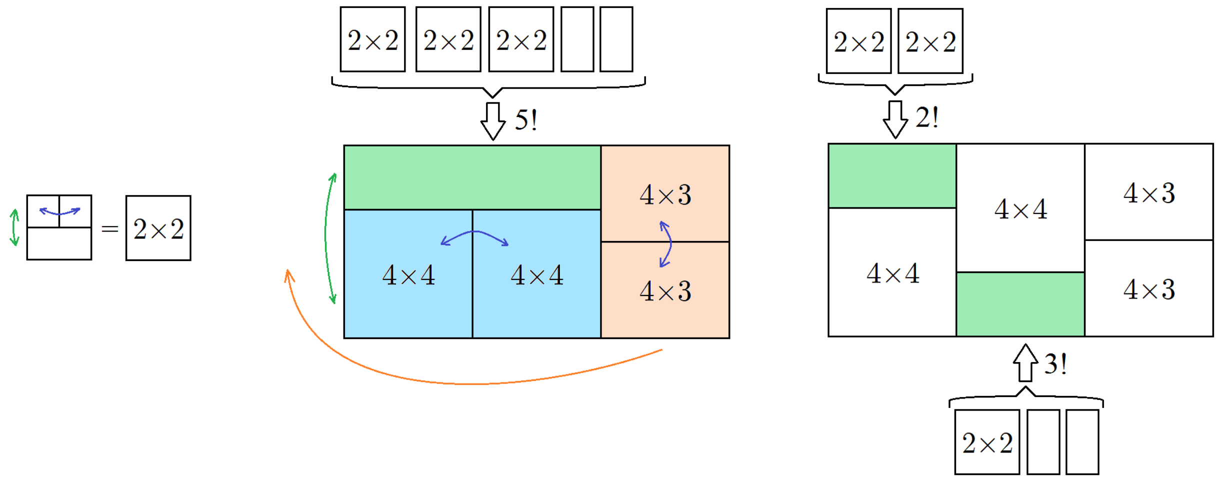
Answer: 16896.

Solution. We will say that two pictures are located in different columns if no block of the first picture is in the same column with any block of the second. It is clear that 4×4 pictures are in different columns with each other and with 4×3 pictures in any arrangement. Thus, the pictures 4×3 will necessarily be strictly one under the other. Both 4×4 paintings are pressed to the floor or ceiling, since at least 6 columns must contain 2 free neighboring cells for the remaining paintings of height 2. There are $3 \cdot 2^4 = 48$ ways to arrange 4 wide paintings in this way (3 ways to choose a column with 3×3 paintings, 2^2 ways to choose "floor/ceiling" in other columns, 2 ways to rearrange pictures 4×4 and 2 more ways to rearrange pictures 4×3). In 16 cases of these 48, an empty section 8×2 remains (there are 4 "degrees of freedom", with 2 options each, they are shown with colored arrows in the middle figure). In the remaining 32 cases, two separate segments 4×2 remain.

Note that in any case, pictures 1×1 together with picture 2×1 will form a square 2×2, so we can replace them with one glued picture (and multiply the answer for the new set of pictures by 4 ways to divide it).

If there is a zone 8×2, then it must be filled with five vertical blocks in some order, there are $5! = 120$ ways to do this.

Let two separate parts 4×2 remain free. Then one part should be divided into two pictures 2×2, and the second one into a picture 2×2 and two pictures 1×2. There are two ways to choose where which part is, then 3 ways to choose which of the 2×2 pictures (including the composite) will be in the second part; after that, $2!$ ways to order blocks for the first section and $3!$ for the second. Totally $2 \cdot 3 \cdot 6 \cdot 2 = 72$ ways.



Totally $4 \cdot (32 \cdot 72 + 16 \cdot 120) = 16896$ options.

Criteria. It is shown that the three smallest paintings (two 1×1 and one 1×2) can only be put together in the form of a square 2×2 — 1 point.

It is shown that all the paintings of width 4 are arranged in 48 ways — 2 more points (if there is an error in the calculation, then +1 point).

It is shown that in the case of two seperate empty zones 4×2 there are 72 ways to arrange small pictures — 2 more points (if there is an error in the calculation, then +1 point).

It is shown that in the case of a single empty space 8×2, there are 120 ways to arrange small paintings — 2 more points (if there is an error in the calculation, then +1 point).

6. There are 17 islands in a kingdom far far away, and each of the islands is inhabited by 119 people. The inhabitants of the kingdom are divided into two castes: knights, who always tell the truth, and liars, who always lie. During a population census, each person was first asked: “Excluding you, is your island inhabited by equal numbers of knights and liars?”. It turned out that on 7 islands everyone answered “Yes”, and on the other islands, everyone answered “No”. Then each person was asked: “Is it true that, including you, the people of your caste contain less than half of the population of your island?”. This time, on some 7 islands, everyone answered “No”, and on the other islands, everyone answered “Yes”. How many liars are in the kingdom? (*P. Mulenko*)

Answer: 1013.

Solution.

1) Consider the first question. On the one hand, the answer «yes» to it will be given either by a knight on the island where exactly 60 knights are living, or by a liar on the island with another number of knights. On the other hand, the answer «no» can be given by either a liar on an island with 59 knights, or by a knight with another number of knights.

   So, on those 7 islands, where everyone answered the first question «yes», there are either 60 or 0 knights; on the remaining 10 islands there are either 59 or 119 knights.

2) The second question, regardless of castes, will be answered «yes» if there are less than half of the knights on the island, and «no» otherwise. So, on those 7 islands, where everyone answered the second question «no», there are at least 60 knights (that is, 60 or 119); and on the remaining 10 islands there are no more than 59 knights (that is, 59 or 0).

3) Let there be 60 knights on x islands and 59 knights on y islands; then (see point 1) there are 0 knights on $7-x$ islands and 119 knights on $10-y$ islands. From point 2 we get: $x+(10-y)=7$, $y+(7-x)=10$. Both equations are equivalent to the equality $y-x=3$.
4) The total number of knights:
$$60\cdot x+59\cdot y+(7-x)\cdot 0+(10-y)\cdot 119=60x+59(x+3)+119(7-x)=59\cdot 3+119\cdot 7=1010.$$

Therefore, there are 1013 liars in the kingdom.

Criteria. All possible numbers of knights and liars on the all islands' types are correctly shown — 2 points.

The problem is solved for the case when the 7 islands in both questions are the same — 2 points.

The problem is solved correctly, but the answer indicates the number of knights, not liars — 6 points.

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Problems for grade R9

Each problem is worth 7 points. Criteria for individual tasks are printed in grey.

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1. Kate and Helen toss a coin. If it comes up heads, Kate wins, if tails, Helen wins. The first time the loser paid the winner 1 dollar, the second time — 2 dollars, then 4, and so on (each time the loser pays 2 times more than in the previous step). After 12 games, Kate became 2023 dollars richer than she was at the beginning. How many of these games did she win?
(*L. Koreshkova, A. Tesler*)

Answer: 9 (all except 4, 8 and, 1024).

Solution. We need to select signs in the equality $\pm1\pm2\pm4\pm8\pm\ldots\pm512\pm1024\pm2048=2023$. If all signs are pluses, then the result is $2^0+\ldots+2^{11}=2^{12}-1=4095$, so we need to replace the pluses with minuses before numbers with total sum $\frac{4095-2023}{2}=1036$. There is only one such set of numbers (because binary representation of a number is unique): $1036=1024+8+4$.

Criteria. 1 point if it is stated that the total amount of money is 4095;

1 more point if it is found that Kate won 3059;

2 points for the correct answer with the example;

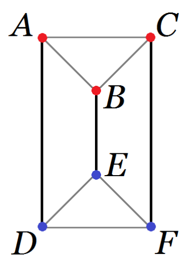
-2 points if there is no proof that the representation of 1036 is unique.

2. On an island, there are several cities connected by roads so that a tourist can get from any city to any other. It turned out that if you close any two cities for quarantine and block all the roads leading to them, you can still drive from any of the remaining cities to any other.
A tourist has randomly chosen three roads, no two of which lead to the same city, and wants to travel along these roads, starting and ending his route in the same city, without visiting any of the cities twice along the way. Can he always do it? (*E. Golikova*)

Solution. Not always. Let us prove that it is impossible for the example shown in the figure (vertical edges are chosen).

1) Whichever two roads we close, the graph will remain connected (after closing one road, it takes one of the two forms shown below; obviously, in each case, closing another road will leave the graph connected).

2) Let's call cities A, B, C red and D, E, F blue. If the tourist does not enter the same city twice, then he must use each selected road exactly once. But whenever he passes through one of the selected roads, the city color changes (and it does not change while using other roads). Since there is an odd number of selected roads, at the end of the path the tourist will end up in a city of a different color than the initial one.



Criteria. 3 points for an example, 4 points for checking that it matches the condition.

3. Solve the equation: $[20x+23]=20+23x$. Recall that $[a]$ denotes the integer part of a number, that is, the largest integer not exceeding a. (*L. Koreshkova*)

Solution. Denote both sides of the equation by n, which is an integer. Then $x = \frac{n-20}{23}$ and $n \leqslant 20x + 23 < n + 1$. It means that

$$n \leqslant \frac{20(n-20)}{23} + 23 < n + 1,$$

that is

$$n \leqslant 43 < n + 7\frac{2}{3}.$$

It turns out that $x = 1 - \frac{k}{23}$ for an arbitrary integer $0 \leqslant k \leqslant 7$.

Answer: $\frac{16}{23}, \frac{17}{23}, \frac{18}{23}, \frac{19}{23}, \frac{20}{23}, \frac{21}{23}, \frac{22}{23}, 1.$

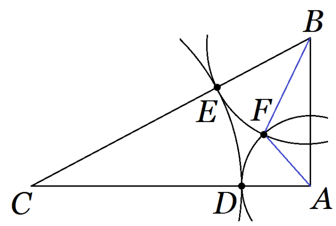
Criteria. 0 points for a solution with result $x = 1$.

2 points if it is obtained that x has the form $1 - \frac{k}{23}$.

-1 point for each missing or extra answer.

4. Given a right triangle ABC with right angle A. A point D divides the side AC in proportion $AD : DC = 1 : 3$. Circles Γ_1 and Γ_2, with centers A and C respectively, both pass through the point D. Γ_2 intersects the hypotenuse at point E. The circle Γ_3 with center B and radius BE intersects Γ_1 inside the triangle at a point F. It turned out that $\angle AFB$ is a right angle. Find BC if $AB = 5$. (*P. Mulenko*)

Answer: 13.



Solution. Denote $AC = x$. Then $AD = x/4$, $DC = CE = 3x/4$, $BE = BC - CE = \sqrt{x^2 + 25} - 3x/4$. By condition, $\angle AFB = 90°$, so $AF^2 + FB^2 = AB^2$, $AD^2 + BE^2 = 25$, so $13x = 12\sqrt{x^2 + 25}$. Squaring both sides, we get $x^2 = 144$, thus $x = 12$, and $BC = 13$.

5. Six cards are given, on which digits 1, 2, 4, 5, 8, and a decimal point are written. All possible numbers are made up of them (each card must be used exactly once, the point cannot be at the beginning or at the end of the number). What is the arithmetic mean of all such numbers? (*M. Karlukova*)

Solution. Note that there are 480 numbers in total ($5! = 120$ ways to arrange the digits and 4 ways to add the decimal point). The sum of these digits is 20, that is, the average is 4; so the sum of all possible 120 numbers without a point is $4 \cdot 11111 \cdot 5! = 11111 \cdot 480$, and the sum of all numbers with all decimal places is $11111 \cdot 480 \cdot 0.1111$. The result is $11111 \cdot 0.1111 = 1234.4321$.

Criteria. 2 points if the number of numbers is found.

-1 points for an arithmetic error in the last step, -3 points for an arithmetic error in the calculation elsewhere.

6. Maria marked points $A(0, 0)$ and $B(1000, 0)$ on the coordinate plane, as well as points $C_1(1, 1)$, $C_2(2, 1), \ldots, C_{999}(999, 1)$. Then she drew all lines AC_i and BC_i $(1 \leqslant i \leqslant 999)$. How many integer points of intersection do all these lines have? (A point is called integer if both coordinates are integers.) (*O. Pyayve*)

Solution. Denote by a_n and b_n the lines passing through A and B, respectively, and also through a point on l whose abscissa is n greater than the abscissa of A (where $1 \leqslant n \leqslant 999$). The lines a_n and a_m for $n < m$ intersect in A, while b_n and b_m in B. The lines a_n and b_m for $n > m$ intersect at a non-integer point (between AB and l). Finally, the lines a_n and b_m for $n \leqslant m$ intersect at a point at a distance k from AB such that $1000(k-1) = k(m-n)$ (it is integer if and only if k is an integer). So k is a divisor of 1000, and vice versa, for each divisor the corresponding $m-n$ is an integer. For each of them there are $\frac{1000}{k}-1$ matching pairs (n,m), so the answer is $1+2+4+8+5+10+20+40+25+50+100+200+125+250+500+1000-16+2=2326$.

Criteria. 2 points for the equation $1000(k-1)=k(m-n)$;

5 points for the statement about $\frac{1000}{k}-1$ pairs of kind (n,m) or something equivalent;

-1 point for miscalculation in the last step.

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Problems for grade R10

Each problem is worth 7 points. Criteria for individual tasks are printed in grey.

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1. Find the sum of all roots of the equation:

$$\sqrt{2x^2 - 2024x + 1023131} + \sqrt{3x^2 - 2025x + 1023132} + \sqrt{4x^2 - 2026x + 1023133} =$$
$$= \sqrt{x^2 - x + 1} + \sqrt{2x^2 - 2x + 2} + \sqrt{3x^2 - 3x + 3}.$$

(*L. Koreshkova*)

Solution. Note that the radical expressions on the left side are obtained from the corresponding radical expressions on the right side by adding $x^2 - 2023x + 1023130 = (x - 1010)(x - 1013)$. Since all radical expressions are positive (it is enough to check for $x^2 - x + 1$ and for $2x^2 - 2024x + 1023131 = 2(x - 506)^2 + 511059$), then the left side is less than the right for $1010 < x < 1013$ and more for $x \notin [1010, 1013]$. Equality is only achieved when $x = 1010$ and $x = 1013$, so the answer is 2023.

Criteria. Not more than 2 points if the answer is found but it is not proved that the equation has no other roots.

2. There are 8 white cubes of the same size. Marina needs to paint 24 sides of the cubes blue and the remaining 24 sides red. Then Kate glues them into a cube $2 \times 2 \times 2$. If there are as many blue squares on the surface of the cube as red ones, then Kate wins. If not, then Marina wins. Can Marina paint faces so that Kate can't reach her goal? (*L. Koreshkova*)

Answer: no.

Solution. Let Marina somehow paint the cubes, and Kate somehow make a cube out of them. Let there be a blue and $24 - a$ red faces on the surface of the cube. Using the idea of the so-called discrete continuity, we will show that Kate can gradually bring the cube to the form she needs. Note that each of the 8 cubes can be rotated so that all of its faces that were outside are inside, and vice versa. If you do this with all eight cubes, then exactly all the faces that were originally inside will appear on the surface, that is, $24 - a$ blue and a red. Note now that each cube can be rotated gradually, so that in one turn two outer faces remain in place and only the third is replaced by the opposite one. With such a rotation, the number of blue faces on the surface changes by no more than 1. So, initially there are a blue squares, at the end their number becomes $24 - a$, and with each action it changes by no more than 1. Since the number 12 is between a and $24 - a$, then at some point there were exactly 12 of them.

Criteria. 2 points are awarded for the idea of turning the cubes one by one.

3. John's favorite TV show is *Couch Lottery*. During the game, viewers can send SMS messages with three-digit numbers containing only digits 1, 2, 3 and 4. At the end of the game, the presenter calls a three-digit number, also consisting only of these digits. An SMS is considered winning if the number in it differs from the presenter's number by no more than one digit (for example, if the presenter named 423, then messages 443 and 123 are winning, but 243 and 224

are not).
John wants to send as few messages as possible so that at least one of them is winning. How many SMS should he send? (*L. Koreshkova*)

Answer: 8.

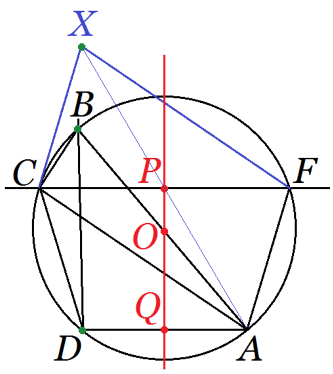
Solution. An example of eight suitable SMS: 111, 122, 212, 221, 333, 344, 434, 443. Indeed, whatever number the presenter names, it contains either at least two digits from the set $\{1, 2\}$, or at least two digits from the set $\{3, 4\}$. If the third digit is from another set, then we replace it with a digit from the same set as the other two, and so that the sum of the digits is odd; so one of the indicated options will definitely be obtained.

Let us now assume that there are less than 8 messages. Then at most one message starts with a certain digit (i. e. with 1). Without loss of generality, let this message (if it exists) be 111. Now consider the options 122, 123, 124, 132, 133, 134, 142, 143, 144. There are 9 of them, and a separate message is needed for each of them (since the first digit is not 1, the second and third must be named correrctly).

Criteria. Estimation gives 4 points, an example – 3 points. For the statement that at least 7 SMS is needed, 1 point is given.

4. An inscribed quadrilateral $ABCD$ with right angle ADB is given. A line $l \parallel AD$ is drawn through the point C. A point F is chosen on l so that $\angle BAF$ is equal to the acute angle between the diagonals AC and BD, and F and C are on different sides of AB. A point X is such that $FXCA$ is a parallelogram. Prove that X lies on BD. (*O. Pyayve*)

Solution. First, note that F lies on the circumcircle of $ABCD$. Indeed, F and D lie on the opposite sides of AC, and $\angle DAF + \angle DCF = \angle DAB + \angle BAF + \pi - \angle CDA = \angle DAC + \angle BAF + \frac{\pi}{2} = \pi$. Denote by P the midpoint of CF, by Q the midpoint of AD, by O the circumcenter. It is clear that they lie on the same straight line (perpendicular bisector to AD). The points X, B, D are obtained from them by a homothety with center at A and coefficient 2, so that they also lie on one line.



Criteria. If it is assumed that B lies between X and D or X lies between B and D, then 3 points are deducted.

5. Solve the equation $a^b + a + b = b^a$ in prime numbers. (*O. Pyayve, P. Mulenko*)

Answer: $a = 5, b = 2$.

Solution. Move b to the right side and group: $a(a^{b-1} + 1) = b(b^{a-1} - 1)$. Since the numbers a and b are prime and obviously different, a is not divisible by b, whence $a^{b-1} + 1 \,\vdots\, b$, which means that $a^{b-1} \equiv -1 \pmod{b}$. On the other hand, according to Fermat's Little Theorem, $a^{b-1} \equiv +1 \pmod{b}$. This means that $-1 \equiv +1 \pmod{b}$ or $2 \equiv 0 \pmod{b}$, i.e. $b = 2$.

We have the equation $a^2 + a + 2 = 2^a$. By direct substitution, we obtain that $a \neq 2$, $a \neq 3$; $a = 5$ matches $(5^2 + 5 + 2 = 32 = 2^5)$. And for any positive integer $a > 5$, this equation has no solutions, because as a increases by one, the left-hand side increases by less than two times: $\big((a+1)^2 + (a+1) + 2\big) - 2(a^2 + a + 2) = (a^2 + 3a + 4) - (2a^2 + 2a + 4) = -a^2 + a = -a(a-1) < 0$.

Note. The problem can be solved for arbitrary positive integers a and b (see problem 5 for grade R11).

Criteria. 2 points for the guessed answer.

6. There are 28 sweets on a table. Peter finds some of them delicious. In one move, Alex can indicate any set of sweets and ask Peter how many of them are delicious. How can Alex find all delicious sweets... (a) in 21 moves; (b) in 20 moves? (*A. Tesler, E. Voronetsky*)

Solution. **a)** Divide the sweets into 7 groups of 4 pieces. In 3 moves we can find out everything about the given 4 candies a, b, c, d, e. g. by asking about sets $\{a, c, d\}$, $\{b, c, d\}$, $\{a, b, c\}$.
If the answers to the first two questions are different, then we find out what candies a and b are, and from the third question we understand what c is. Returning to the first question we know about d.
If the answers to the first two questions are the same, then a and b are the same. If the answer to the third question is less than 2, then a and b are not delicious, and if 2 or more, then they are delicious; by parity of this answer we determine whether c is delicious. And again, returning to the first question, we determine what d is.

Note. Another set of questions would also work, such as $\{a, c\}$, $\{b, c\}$, $\{a, b, d\}$.

b) There are different solutions for this part. Let us give a solution that allows find all sweets even in 19 moves. Let us prove the following assertion: "If for $n > 0$ sweets the problem can be solved in m questions, then for $n + 3$ sweets it can be solved in $m + 2$ questions". (Since one question is enough for one sweet, $1 + 9 \cdot 2 = 19$ questions are enough for $28 = 1 + 9 \cdot 3$ sweets.)

Indeed, let there be a way to handle n sweets in m questions, and let the first of these questions be asked about some set X. Let's add three sweets a, b, c, then add three questions about the sets $\{a, c\} \cup X$, $\{b, c\} \cup X$, $\{a, b, c\}$ to the list of questions, and *remove* the question about X. By answering these questions, we can find out what sweets a, b, c are and how many delicious sweets are in X (exactly the same way as in part a but d is replaced by X).

Criteria. 2 points for part (a) and 5 points for (b).

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Problems for grade R11

Each problem is worth 7 points. Criteria for individual tasks are printed in grey.

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1. Find the sum of all roots of the equation:

$$\sqrt{2x^2-2024x+1023131}+\sqrt{3x^2-2025x+1023132}+\sqrt{4x^2-2026x+1023133}=$$
$$=\sqrt{x^2-x+1}+\sqrt{2x^2-2x+2}+\sqrt{3x^2-3x+3}.$$

(*L. Koreshkova*)

Solution. Note that the radical expressions on the left side are obtained from the corresponding radical expressions on the right side by adding $x^2-2023x+1023130=(x-1010)(x-1013)$. Since all radical expressions are positive (it is enough to check for x^2-x+1 and for $2x^2-2024x+1023131=2(x-506)^2+511059$), then the left side is less than the right for $1010<x<1013$ and more for $x\notin[1010,1013]$. Equality is only achieved when $x=1010$ and $x=1013$, so the answer is 2023.

Criteria. Not more than 2 points if the answer is found but it is not proved that the equation has no other roots.

2. There are 8 white cubes of the same size. Marina needs to paint 24 sides of the cubes blue and the remaining 24 sides red. Then Kate glues them into a cube $2\times2\times2$. If there are as many blue squares on the surface of the cube as red ones, then Kate wins. If not, then Marina wins. Can Marina paint faces so that Kate can't reach her goal? (*L. Koreshkova*)

Answer: no.

Solution. Let Marina somehow paint the cubes, and Kate somehow make a cube out of them. Let there be a blue and $24-a$ red faces on the surface of the cube. Using the idea of the so-called discrete continuity, we will show that Kate can gradually bring the cube to the form she needs. Note that each of the 8 cubes can be rotated so that all of its faces that were outside are inside, and vice versa. If you do this with all eight cubes, then exactly all the faces that were originally inside will appear on the surface, that is, $24-a$ blue and a red. Note now that each cube can be rotated gradually, so that in one turn two outer faces remain in place and only the third is replaced by the opposite one. With such a rotation, the number of blue faces on the surface changes by no more than 1. So, initially there are a blue squares, at the end their number becomes $24-a$, and with each action it changes by no more than 1. Since the number 12 is between a and $24-a$, then at some point there were exactly 12 of them.

Criteria. 2 points are awarded for the idea of turning the cubes one by one.

3. Kate and Helen toss a coin. If it comes up heads, Kate wins, if tails, Helen wins. The first time the loser paid the winner 1 dollar, the second time — 2 dollars, then 4, and so on (each time the loser pays 2 times more than in the previous step). At the beginning, Kate had a one-digit number of dollars, and Helen had four-digit number. But at the end, Helen's capital became

two-digit while Kate's became three-digit. What minimum number of games could Kate win? Capitals cannot go negative during the game. (*L. Koreshkova, A. Tesler*)

Solution. Denote by n the amount of money by which Kate became richer (and Helen poorer). Note that Kate won the last game (otherwise she would have lost more money than she gained at all previous stages). This means that the sequence of games can be divided into series, in each of which Kate won the last game and lost all the others (a series can also consist of one game). If the series started with game number k and ended with game number m, then Kate won $-2^k - 2^{k+1} - \ldots - 2^{m-2} + 2^{m-1} = 2^k$ dollars. Thus, the binary representation of n uniquely describes the set of games Kate won (except for the number of the last game): a summand 2^k means that the next series started with the game number $k+1$, i. e. the game Kate won the game number k.

By the condition, $901 \leqslant n \leqslant 998$. But all numbers from 901 to 998 contain $2^7 + 2^8 + 2^9$ in binary representation, so Kate won the seventh, eighth, ninth game, as well as the last one (its number is greater than 9, otherwise there would be no summand 2^9) — already at least 4 games.

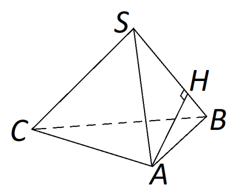
In addition, in the first 6 games, Kate had to win at least 3 times:
1) she won at least one of the first four games because $9 - 1 - 2 - 4 - 8 < 0$;
2) at least one of the next two is also won since $9 \pm 1 \pm 2 \pm 4 \pm 8 - 16 - 32 < 0$; 3) if only one of the first four games is won, then after them the sum is no more than 10, and the fifth and sixth must be won.

Thus Kate won at least 7 games. Here is an example for 7 games: initially Kate has 9 dollars and Helen has — 1000 rubles, and 10 games are played. Then $n = 985 = 2^0+2^3+2^4+2^6+2^7+2^8+2^9 = (-2^0 - 2^1 + 2^2) + (2^3) + (2^4) + (-2^5 + 2^6) + (2^7) + (2^8) + (2^9)$, so Kate won games with numbers $3, 4, 6, 7, 8, 9, 10$, and Helen won games $1, 2, 5$. At the end, Kate has 994 dollars, and Helen has 15 dollars.

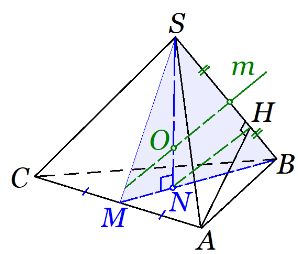
Answer: 7 games.

Criteria. 5 points are given for the estimation (including 1 point for the fact that at least 10 games have passed), and 2 points for an example.

4. A regular pyramid $SABC$ (with base ABC) with the height AH of face SAB is drawn on the plane using orthogonal projection, as shown in the picture. How to construct the image of the circumcenter of the pyramid using a compass and a ruler? (*A. Tesler*)



Solution. Let M be the midpoint of AC, and N the center of $\triangle ABC$. Then the center of the circumscribed sphere lies on SN (since the pyramid is regular). The projection of M is constructed as the midpoint of the projection of AC, and the projection of N as a point dividing the projection of BM in the ratio $2:1$.



Denote by m the line parallel to MH and passing through the midpoint of SB. It passes through the center of the circumscribed sphere: AH and CH are perpendicular to SB, so m is perpendicular to SB, and m also intersects SN. The projection of m is constructed as a parallel translation of the projection of MH passing through the middle of the projection of

SB. This projection intersects SN exactly in the projection of the center of the circumscribed sphere.

5. Solve the equation $a^b + a + b = b^a$ in positive integers. (*O. Pyayve, E. Voronetsky*)

Answer: $a = 5, b = 2$.

Solution. If $a = 1$ or $b = 1$, then there are no solutions.

If $b = 2$, then we get $2^a = a^2 + a + 1$. For $a < 5$ there are no solutions, $a = 5$ is fine, and for $a \geqslant 5$ the left side increases by less than 2 times when 1 is added to a.

Let $b \geqslant 3$. Then

$$a^b + a + b \leqslant a^b + ab \leqslant a^b + ba^{b-2} < \left(a + \frac{1}{a}\right)^b$$

(the last inequality follows from the Newton binomial expansion for $\left(a + \frac{1}{a}\right)^b$). So

$$\left(a + \frac{1}{a}\right)^b > b^a > a^b,$$

that is

$$\frac{\ln\left(a + \frac{1}{a}\right)}{a} > \frac{\ln b}{b} > \frac{\ln a}{a}.$$

Note that $f(a) = \frac{\ln(a)}{a}$ decreases as $a \geqslant 3$ and $f(2) = f(4)$ (the derivative of this function is equal to $f'(a) = \frac{1 - \ln a}{a^2}$, and it is negative for $a \geqslant e$). Therefore, there are no solutions with $a = 2$, $b \geqslant 4$ and with $b \geqslant a \geqslant 3$.

On the other hand, one can check that

$$\frac{\ln\left(a + \frac{1}{a}\right)}{a} < \frac{\ln(a-1)}{a-1}$$

for $a \geqslant 4$. Indeed, for $a = 4$ it is

$$\left(4 + \frac{1}{4}\right)^3 = 64 + 12 + \frac{3}{4} + \frac{1}{64} < 81 = 3^4,$$

and the derivative of the expression $g(a) = a \cdot \ln(a-1) - (a-1) \cdot \ln(a + 1/a)$ is

$$g'(a) = \frac{1}{a-1} + \frac{a^2 + 2a - 1}{a^3 + a} - \ln 1 + \frac{a+1}{a^2 - a}.$$

But

$$\ln 1 + \frac{a+1}{a^2 - a} < \frac{a+1}{a^2 - a},$$

So

$$g'(a) > \frac{a^3 - 3a}{a(a-1)(a^2+1)} > 0$$

already for $a \geqslant 3$.

Thus the equation has no solutions for $a \geqslant 4$.

Note. Instead of evaluating for $(a + 1/a)^b$, the one for $(a+1)^b$ (valid for $b = 2$) can be used, it simplifies calculations, but you need to consider much more exceptions.

Criteria. 1 point is given for the correct answer. An argument showing that at least one variable is bounded gives 3 points.

6. There are 28 sweets on a table. Peter finds some of them delicious. In one move, Alex can indicate any set of sweets and ask Peter how many of them are delicious. How can Alex find all delicious sweets... (a) in 21 moves; (b) in 20 moves? (*A. Tesler, E. Voronetsky*)

Solution. **a)** Divide the sweets into 7 groups of 4 pieces. In 3 moves we can find out everything about the given 4 candies a, b, c, d, e. g. by asking about sets $\{a, c, d\}$, $\{b, c, d\}$, $\{a, b, c\}$.
If the answers to the first two questions are different, then we find out what candies a and b are, and from the third question we understand what c is. Returning to the first question we know about d.
If the answers to the first two questions are the same, then a and b are the same. If the answer to the third question is less than 2, then a and b are not delicious, and if 2 or more, then they are delicious; by parity of this answer we determine whether c is delicious. And again, returning to the first question, we determine what d is.

Note. Another set of questions would also work, such as $\{a, c\}$, $\{b, c\}$, $\{a, b, d\}$.

b) There are different solutions for this part. Let us give a solution that allows find all sweets even in 19 moves. Let us prove the following assertion: "If for $n > 0$ sweets the problem can be solved in m questions, then for $n + 3$ sweets it can be solved in $m + 2$ questions". (Since one question is enough for one sweet, $1 + 9 \cdot 2 = 19$ questions are enough for $28 = 1 + 9 \cdot 3$ sweets.)

Indeed, let there be a way to handle n sweets in m questions, and let the first of these questions be asked about some set X. Let's add three sweets a, b, c, then add three questions about the sets $\{a, c\} \cup X$, $\{b, c\} \cup X$, $\{a, b, c\}$ to the list of questions, and *remove* the question about X. By answering these questions, we can find out what sweets a, b, c are and how many delicious sweets are in X (exactly the same way as in part a but d is replaced by X).

Criteria. 2 points for part (a) and 5 points for (b).