International Mathematical Olympiad
«Formula of Unity» / «The Third Millennium»
Year 2022/2023. Qualifying round

Problems for grade R5

Each task is assessed at 7 points. Some problems have their own criteria (printed in gray).

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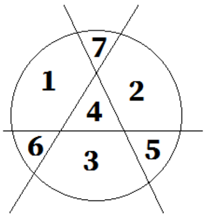
1. A circle is divided into 7 parts by 3 lines. Is it possible to write the numbers from 1 to 7 into these parts (one number in each part) so that the sum of numbers on one side of each line is equal to the sum of numbers on the other side?

   Solution. Yes:



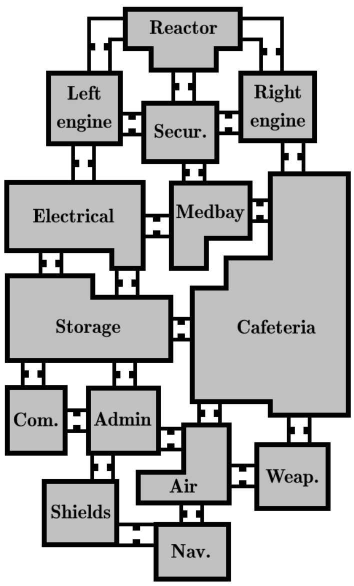
   Criteria. The correct example — 7 points. An example that does not satisfy any condition of the problem (e.g. using only numbers from 1 to 7, but with repetitions) — 0 points.

2. To participate in the Olympiad, Marina needs to buy a notebook, a pen, a ruler, a pencil and an eraser. If she buys a notebook, a pencil and an eraser, she will spend 47 tugriks. If she buys a notebook, a ruler and a pen, she will spend 58 tugriks. How much money will she need for the whole set if the notebook costs 15 tugriks?

   Solution. If Marina buys both sets given, she spends $47 + 58 = 105$ tugriks, but she gets an extra notebook, so the full set costs $105 - 15 = 90$ tugriks.

   Criteria. Only the answer is given without any explanation — 1 point. If student guesses prices of the pen and/or the pencil (although it is not said that prices are integers) — 0 points.

3. A research spacecraft has a reactor failure and some poisonous substances leak from the reactor. All corridors between rooms are equipped with airtight doors, but there is no time to close individual doors. However, the captain can give the command «Close N doors», after which the ship's artificial intelligence will close random N doors. What is the smallest N to guarantee that the whole team can survive in the cafeteria?



   Solution. There are 23 corridors in total in the spacecraft. If no more than 21 doors are closed, it is possible that the corridors between the reactor and the right engine and between the right engine and the cafeteria remain open, which puts the team in danger. Therefore, it is necessary to close at least 22 doors.

   Criteria. Only the answer is given without any explanation — 0 points. Any mistake in counting the number of corridors — 4 points.

4. A school was opened on the island of knights and liars (a knight always tells the truth, a liar always lies). All $2N$ students are of different heights. They stood in a circle and everyone said: "I am taller than the student standing in front of me!" How many knights are there in the school?

Solution. In each pair of students one of the two is actually higher, so he tells the truth and is a knight. Therefore N knights in total.

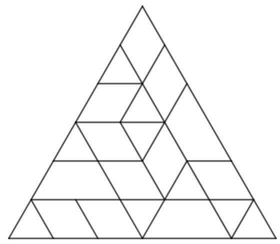
Criteria. The fact of precisely one knight in each pair is not proved in any way — 1 point. One or few examples are given as a proof — 0 points.

5. Kate wrote a number divisible by 5 on a board and encrypted it according to the rules of alphametic puzzles (different letters correspond to different digits, the same letters — to the same digits). She got the word "GUATEMALA". How many different numbers could Kate write on the board?

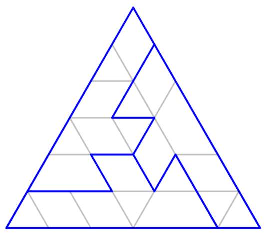
   Solution. The number must be divisible by 5, so the letter «A» is 0 or 5. If it is 0, then for the remaining letters («G», «U», «T», «E», «M», «L») there are $9!/(9-6)!$ variants; if «A» is 5, then for the remaining letters there are $8 \cdot 8!/(8-5)!$ variants, since «G» cannot be 0. In total, $9!/6 + 8 \cdot 8!/6 = 114240$ options.

   Criteria. It is explicitly stated that for the letter «A» there are 2 options: 5 or 0 — 2 points. If both options are further solved, then another 5 points. If the answer is given as a correct expression and not calculated — do not subtract any points.

6. Cut the triangle on the picture along the marked lines into three equal parts (the parts are called equal if they match both in shape and size).

   Solution.

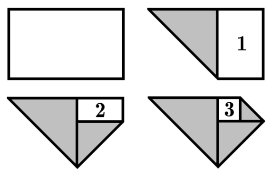
   

7. Three cars A, B and C start simultaneously from the same point of a circular track. A and B travel clockwise, while C — counter-clockwise. All cars move at constant (but pairwise different) speeds. After exactly 7 minutes of the race, A meets C for the first time. 46 minutes later, A and B meet for the first time. How long does it take from the start to the first meeting of all three cars?

   Solution. A and C meet once every 7 minutes, and A and B — once every 53 minutes. So, all together they will meet at such time that is a multiple of both 7 and 53, hence once in $7 \cdot 53 = 371$ minutes.

   Criteria. 46 minutes are used in calculations instead of 53 — 3 points. If student guesses track's length and/or cars' speeds — 1 point. Only the answer is given without any explanation — 0 points.

8. There is a rectangular piece of paper with one side white and the other side grey. It was bent as shown in the picture. The perimeter of the first rectangle is 20 more than the perimeter of the second one. The perimeter of the second rectangle is 16 more than the perimeter of the third one. Find the perimeter of the whole piece of paper.

   Solution. As you can see from the picture, the perimeter of the rectangle decreases by two lengths of the short side with each folding. So the short side of rectangle-1 equals $20/2 = 10$, and the short side of rectangle-2 equals $16/2 = 8$. Hence the long side of rectangle-1 is 18, and the long side of the original sheet is 28. Then the perimeter is $(28 + 18) \cdot 2 = 92$.

   Criteria. If student guesses rectangles' sides in any way (or simply states the lengths of the sides without an explanation) — 0 points.

International Mathematical Olympiad
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Year 2022/2023. Qualifying round

Problems for grade R6

Each task is assessed at 7 points. Some problems have their own criteria (printed in gray).

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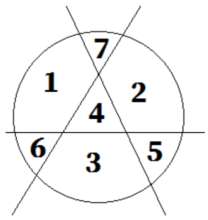
1. A circle is divided into 7 parts by 3 lines. Is it possible to write 7 consecutive positive integers into these parts (one number in each part) so that the sum of numbers on one side of each line is equal to the sum of numbers on the other side?

Solution. Yes:



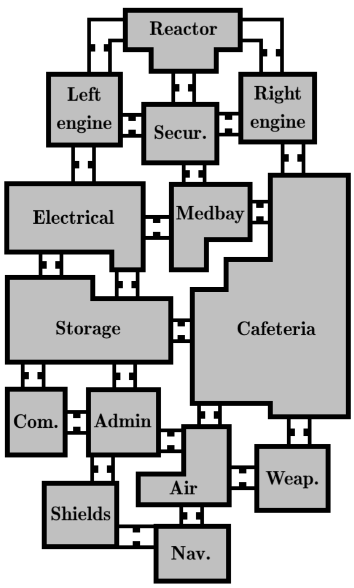
Criteria. The correct example — 7 points. An example that does not satisfy any condition of the problem (e.g. using not only positive consecutive integers) — 0 points.

2. To participate in the Olympiad, Marina needs to buy a notebook, a pen, a ruler, a pencil. If she buys a notebook, a pencil and a ruler, she will spend 47 tugriks. If she buys a notebook, a ruler and a pen, she will spend 58 tugriks. If she buys a pen and a pencil, she will spend 15 tugriks. How much money will she need for the whole set?

Solution. If Marina buys all the three sets given, she spends $47 + 58 + 15 = 120$ tugriks for 2 full sets. Therefore one full set costs $120/2 = 60$ tugriks.

Criteria. Only the answer is given without any explanation — 1 point. If student guesses prices of the pen and/or the pencil (although it is not said that prices are integers) — 0 points.

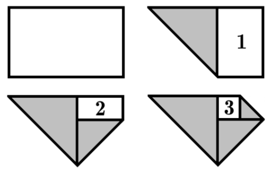
3. A research spacecraft enters an asteroid belt that may damage the ship's hull, causing depressurization. All corridors between rooms are equipped with airtight doors. The captain has an assistant droid that can close (but not open back) the doors in the corridors he passes through. Will the droid be able to close all the doors on the spacecraft?



Solution. There are 23 corridors and 14 compartments in the spacecraft. Every time the droid goes through one of the compartments, it closes two corridors: the one it entered the room through and the one it left through. Therefore, in all the rooms, except for, maybe, two of them, there should be an even number of entrances (those two rooms can become starting and ending points of the droid's route, so there can be an odd number of entrances). However, there are 6 such rooms in the spacecraft (cafeteria, storage, medbay, both engines and reactor), so the droid will not be able to close all the doors.

Criteria. If the solution mentions that the droid follows the Eulerian (unicursal) path or that there can be no more than 2 rooms in it's path with an odd number of doors — 3 points. One or few examples are given as a proof — 0 points.

4. There is a rectangular piece of paper with one side white and the other side grey. It was bent as shown in the picture. The perimeter of the first rectangle is 20 more than the perimeter of the second one. The perimeter of the second rectangle is 16 more than the perimeter of the third one. Find the perimeter of the whole piece of paper.



Solution. As you can see from the picture, the perimeter of the rectangle decreases by two lengths of the short side with each folding. So the short side of rectangle-1 equals $20/2 = 10$, and the short side of rectangle-2 equals $16/2 = 8$. Hence the long side of rectangle-1 is 18, and the long side of the original sheet is 28. Then the perimeter is $(28 + 18) \cdot 2 = 92$.

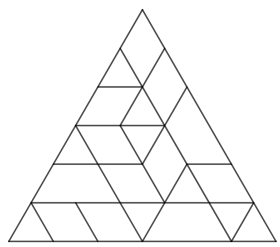
Criteria. If student guesses rectangles' sides in any way (or simply states the lengths of the sides without an explanation) — 0 points.

5. Kate wrote a number divisible by 25 on the board and encrypted it according to the rules of alphametic puzzles (different letters correspond to different digits, the same letters — the same digits). She got the word "GUATEMALA". How many different numbers could Kate write on the board?

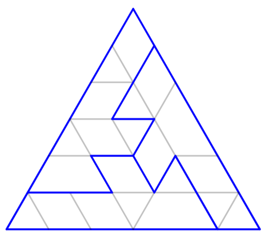
Solution. The number must be divisible by 25, so LA is equal to 25, 50 or 75 (00 is not possible since the letters are different). If $\overline{LA}$ is equal to 50, then for the remaining letters (G, U, T, E, M) there are $8!/(8-5)!$ variants; otherwise for the remaining letters there are $7 \cdot 7!/(7-4)!$ options. In total, $8!/6 + 2 \cdot 7 \cdot 7!/6 = 18480$.

Criteria. It is shown that option 00 is not suitable for the letters LA and all 3 cases are demonstrated — 3 points. If both options are further solved correctly — 4 more points. If the answer is given as a correct expression and not calculated — do not subtract any points.

6. Cut the triangle on the picture along the marked lines into three equal parts (the parts are called equal if they match both in shape and size).



Solution.



7. A school was opened on the island of knights and liars (a knight always tells the truth, a liar always lies). All $2N$ students lined up in pairs one after another (in other words, in two equal columns). The two people standing first said: "I am taller than 2 people: my neighbor in a pair and the person behind me". The last two said: "I am also taller than 2 people: my neighbor in a pair and the person in front of me". Finally, everyone else said: "I am taller than 3 people: my neighbor in a pair, the person in front of me and the person behind me".
a) Find the maximal possible amount of knights among the students.
b) Is it possible for all the students to be liars?

Solution. A) There is no more than one knight in each pair, so there are no more than N knights (this amount can be achieved by placing N higher students in chess order).
B) Yes. If all the students are of the same height, then everyone is lying.

Criteria. Problem a) is fully solved — 5 points. If only one part of the solution is given (either an estimation or an example) — 2 points. One or few examples are given as a proof — 0 points. Problem b) is fully solved — 2 points.

8. Four cars A, B, C and D start simultaneously from the same point of a circular track. A and B travel clockwise, while C and D — counter-clockwise. All cars move at constant (but pairwise different) speeds. After exactly 7 minutes of the race A meets C for the first time, and at the same moment B meets D for the first time. 46 minutes later, A and B meet for the first time. How long does it take from the start to the first meeting of C and D?

   Solution. If A meets C and B meets D every 7 minutes, then their convergence speeds are equal: $V_A + V_C = V_B + V_D$. Therefore, the removal speeds of A with B and C with D are equal: $V_A - V_B = V_D - V_C$. So, since A meets B for the first time at the 53^{rd} minute, then C and D will meet for the first time at the same moment.

   Criteria. 46 minutes are used in calculations instead of 53 — 3 points. If it is stated that any of cars' speeds are equal — 1 point. Only the answer is given without any explanation — 0 points.

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Problems for grade R7

Each task is assessed at 7 points. Some problems have their own criteria (printed in gray).

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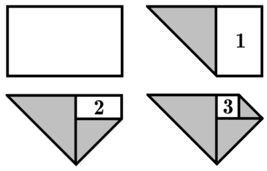
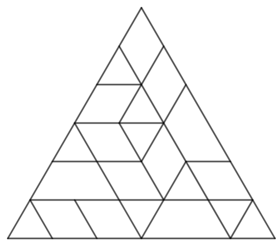
1. There is a rectangular piece of paper with one side white and the other side grey. It was bent as shown in the picture. The perimeter of the first rectangle is 20 more than the perimeter of the second one. The perimeter of the second rectangle is 16 more than the perimeter of the third one. Find the area of the whole piece of paper.



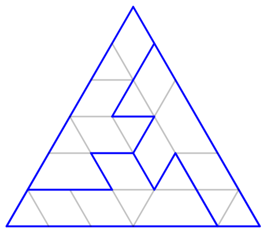
Solution. As you can see from the picture, the perimeter of the rectangle decreases by two lengths of the short side with each folding. So the short side of rectangle-1 equals $20/2 = 10$, and the short side of rectangle-2 equals $16/2 = 8$. Hence the long side of rectangle-1 is 18, and the long side of the original sheet is 28. Then the area equals $28 \cdot 18 = 504$.

Criteria. The perimeter of the rectangle is given as an answer instead of area — 3 points. If student guesses rectangles' sides in any way (or simply states the lengths of the sides without an explanation) — 0 points.

2. Cut the triangle on the picture along the marked lines into three equal parts (the parts are called equal if they match both in shape and size).



Solution.



3. Kate wrote a number divisible by 8 on a board and encrypted it according to the rules of alphametic puzzles (different letters correspond to different digits, the same letters — the same digits). She got the word "GUATEMALA". How many different numbers could Kate write on the board?

Solution. In order for a number to be divisible by 8, «ALA» must be divisible by 8 with «A» to be an even digit. «ALA» $= 101 \cdot$ «A» $+ 10 \cdot$ «L» $=$ $(100 \cdot$ «A» $+ 8 \cdot$ «L»$)$ $+$ «A» $+ 2 \cdot$ «L». The part in brackets is obviously divisible by 8, so it is enough to require for («A» $+ 2 \cdot$ «L») be divisible by 8. There are 11 options for that to happen: (0,4), (0,8), (2,3), (2,7), (4,2), (4,6), (6,1), (6,5), (6,9), (8,0), (8,4). In three of them with zero for the remaining five letters («G», «U», «T», «E», «M») there are $8!/(8-5)!$ options; in the other eight — $7 \cdot 7!/(7-4)!$. In total, $3 \cdot 8!/3! + 8 \cdot 7 \cdot 7!/3! = 67200$.

Criteria. The criteria of divisibility by 8 is said and it is clearly indicated that «A» is even — 1 point. It is proved that «A» $+ 2$«L» or 5«A» $+ 10$«L» is divisible by 8 — 3 points. For each lost case for «A» and «L» 1 point sould be deducted. If cases with 0 are calculated in the same way as others — 2 points should be subtracted.

4. Four cars A, B, C and D start simultaneously from the same point of a circular track. A and B travel clockwise, while C and D — counter-clockwise. All cars move at constant (but pairwise different) speeds. After exactly 7 minutes of the race A meets C for the first time, and at the same moment B meets D for the first time. 46 minutes later, A and B meet for the first time. How long does it take from the start to the first meeting of all four cars?

Solution. A and C meet once every 7 minutes, and A and B — once every 53 minutes. So, all together they will meet at such time that is a multiple of both 7 and 53, hence once in $7 \cdot 53 = 371$ minutes. From the other hand, B meets D every 7 minutes as well, so at the 371^{st} minutes D will be at the same place as the rest 3 cars.

Criteria. 46 minutes are used in calculations instead of 53 — 3 points. If it is stated that any of cars' speeds are equal — 1 point. Only the answer is given without any explanation — 0 points.

5. The squares of the first 2022 natural numbers are written in a row: $1, 4, 9, \ldots, 4088484$. For each written number, except for the first and the last ones, the arithmetic mean of its left and right neighbors was calculated and written under it (for example, $\frac{1+9}{2} = 5$ was written under the number 4). For the resulting string of 2020 numbers, we did the same. So we continued until we reached a line in which there are only two numbers. Find these numbers.

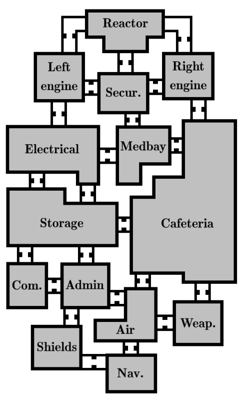
Solution. Let's take a look at one of the numbers x^2. Under it

$$\frac{(x-1)^2 + (x+1)^2}{2} = \frac{x^2 - 2x + 1 + x^2 + 2x + 1}{2} = \frac{2x^2 + 2}{2} = x^2 + 1$$

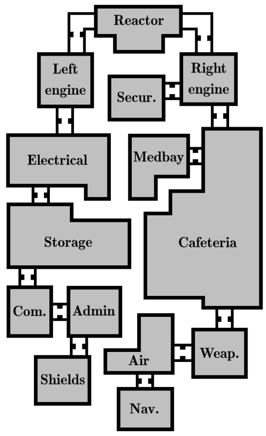
will be written, so each time numbers increase by one. Initially, there are 2022 numbers and each time 2 of them disappears, so there will be 1011 lines of numbers, and in the very last line there will be two central numbers increased by 1010: $1011^2 + 1010 = 1023131$ and $1012^2 + 1010 = 1025154$.

Criteria. It is proved generally that each time every number increases by 1 — 3 points. If the problem is solved correctly without a competent proof of the increment — 4 points. Any mistake in counting the remaining numbers — -2 points.

6. A research spacecraft has a reactor failure and some poisonous substances leak from the reactor. All corridors between rooms are equipped with airtight doors, but there is no time to close individual doors. However, the captain can give the command «Close N doors», after which the ship's artificial intelligence will close random N doors. What is the smallest N to guarantee that at least one of the compartments of the ship will be safe?



Solution. There are 23 corridors and 14 compartments in the spacecraft. It can be interpreted as a graph, where the compartments are vertices and the corridors are edges. In order for such a graph to remain connected (so that there is a path between any two vertices) with a minimum number of edges, it must be a graph-tree, so it must have at least $14 - 1 = 13$ edges. In other words, if you close no more than $23 - 13 = 10$ doors, then it is possible to a way between all the compartments exist(see Fig.), what puts the team in danger. Therefore, it is necessary to close at least $10 + 1 = 11$ doors and hide in those compartments that are completely «cut off» from the reactor (indeed, closing any other corridor in the figure will lead to the «cutting off» part of the compartments from the reactor).



Criteria. It is shown that less than 11 doors will not be enough (e.g. an example is given for 10 doors and it is shown that there will be a passage between all compartments) — 3 points. It is proved that 11 closed doors are enough — 3 points. Any mistake in counting the number of corridors and/or compartments — −2 points.

7. Let us call a positive integer *useful* if its decimal notation contains neither zeroes nor equal digits, and if the product of all its digits is divisible by the sum of these digits. Are there any two consecutive 3-digit useful numbers?

Solution. Yes. For example: 578, 579 or 875, 876.

Criteria. If only the right example is given without justification — 5 points.

8. A school was opened on the island of knights and liars (a knight always tells the truth, a liar always lies). All $2N$ students are of different heights. They lined up in pairs one after another (in other words, in two equal columns). The two people standing first said: "I am taller than 2 people: my neighbor in a pair and the person behind me". The last two said: "I am also taller than 2 people: my neighbor in a pair and the person in front of me". Finally, everyone else said: "I am taller than 3 people: my neighbor in a pair, the person in front of me and the person behind me".
a) Find the largest possible number of knights among the students.
b) Is it possible for all the students to be liars?

Solution. A) There is no more than one knight in each pair, so there are no more than N knights (this amount can be achieved by placing N higher students in chess order).
B) No. All the students are of the different heights, so the heighest of them is always a knight.

Criteria. Problem a) is fully solved — 5 points. If only one part of the solution is given (either an estimation or an example) — 2 points. One or few examples are given as a proof — 0 points. Problem b) is fully solved — 2 points.

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Problems for grade R8

Each task is assessed at 7 points. Some problems have their own criteria (printed in gray).

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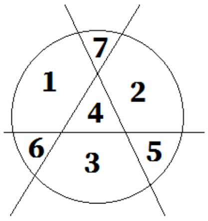
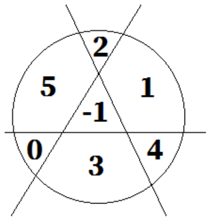
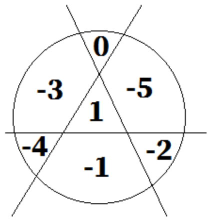
1. A circle is divided into 7 parts by 3 lines. Maria wants to write 7 consecutive integers into these parts (one number in each part) so that the sum of numbers on one side of each line is equal to the sum of numbers on the other side. Find 3 ways to do it which differ with sets of numbers used.

   Solution. Several (not all) examples:



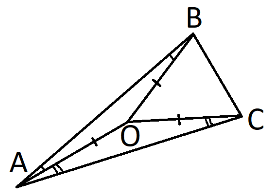




2. *Breaking* of an acute triangle ABC is the operation when a point O such that $OA = OB = OC$ is chosen inside the triangle, and it is cut into triangles OAB, OAC, OBC. Peter took a triangle with angles 3°, 88° and 89° and *broke* it into three triangles. Then he chose one of the pieces (also acute) and *broke* it. So he continued until all the triangles were obtuse. How many triangles did he get in total?

   Solution. The acute angle of an acute-angled triangle doubles each time: after the first cut it is 6°, after the second 12°, then 24°, 48°, 96°. The remaining triangles are always obtuse. So we make 5 breaks, thus there are $1 + 2 \cdot 5 = 11$ triangles in total.

   Consider a breaking of an acute-angled triangle ABC. O is the center of its circumcircle, so it lies inside the triangle, and the triangles AOB, AOC, BOC are isosceles. If $\angle BAO = \alpha, \angle CAO = \beta$, then $\angle AOB = 180^\circ - 2\alpha, \angle AOC = 180^\circ - 2\beta$, whence $BOC = 2\alpha + 2\beta = 2\angle BAC$. For the initial triangle (with angles $\angle A = 3^\circ, \angle B = 88^\circ, \angle C = 89^\circ$), only $\triangle BOC$ is acute (moreover, it is isosceles, $\angle BOC = 6^\circ$).



   Further, if we mark a point O_2 in it, then the triangle O_2BC is acute ($\angle O_2 = 12^\circ$), and the other two triangles are equal and obtuse. Analogously, $\angle O_3 = 24^\circ$, $\angle O_4 = 48^\circ$, $\angle O_5 = 96^\circ$. At this point, all three new triangles are obtuse and the process stops. Thus there were 5 breakings. Each breaking increases the number of triangles by 2, so there are 11 triangles in total.

3. Let us call a positive integer $n > 5$ *new* if there exists an integer which is divisible by all the numbers $1, 2, \ldots, n-1$ but not by n. What is the maximal number of consecutive new integers?

   Answer: 3.

   Solution. Example: number 7 is new (60 is divisible by all the numbers from 1 to 6 but not by 7);

number 8 is new (420 is divisible by all the numbers from 1 to 7 but not by 8);
number 9 is new (840 is divisible by all the numbers from 1 to 8 but not by 9).

Estimation: each fourth number if of type $n = 4k + 2 = 2(2k + 1)$; if a number is divisible by 2 and $2k + 1$, it is also divisible by n, so it cannot be new.

4. The arithmetic mean of several positive integers equals to 20.22. Prove that at least two of the numbers are equal.

   Solution. Since 20.22 is equal to an irreducible fraction with denominator 50, the number of the integers is divisible by 50. However if all 50 numbers are different, then the arithmetic mean of $50n$ distinct natural numbers is at least $25.5n$, thus more than 25.

5. The squares of the first 2022 natural numbers are written in a row: $1, 4, 9, \ldots, 4088484$. For each written number, except for the first and the last ones, the arithmetic mean of its left and right neighbors was calculated and written under it (for example, $\frac{1+9}{2} = 5$ was written under the number 4). For the resulting string of 2020 numbers, we did the same. So we continued until we reached a line in which there are only two numbers. Find these numbers.

   Solution. Let's take a look at one of the numbers x^2. Under it

$$\frac{(x-1)^2 + (x+1)^2}{2} = \frac{x^2 - 2x + 1 + x^2 + 2x + 1}{2} = \frac{2x^2 + 2}{2} = x^2 + 1$$

   will be written, so each time numbers increase by one. Initially, there are 2022 numbers and each time 2 of them disappears, so there will be 1011 lines of numbers, and in the very last line there will be two central numbers increased by 1010: $1011^2 + 1010 = 1023131$ and $1012^2 + 1010 = 1025154$.

   Criteria. It is proved generally that each time every number increases by 1 — 3 points. If the problem is solved correctly without a competent proof of the increment — 4 points. Any mistake in counting the remaining numbers — -2 points.

6. Four cars A, B, C and D start simultaneously from the same point of a circular track. A and B travel clockwise, while C and D — counter-clockwise. All cars move at constant (but pairwise different) speeds. After exactly 7 minutes of the race A meets C for the first time, and at the same moment B meets D for the first time. 46 minutes later, A and B meet for the first time. How long does it take from the start to the first meeting of C and D?

   Solution. If A meets C and B meets D every 7 minutes, then their convergence speeds are equal: $V_A + V_C = V_B + V_D$. Therefore, the removal speeds of A with B and C with D are equal: $V_A - V_B = V_D - V_C$. So, since A meets B for the first time at the 53^{rd} minute, then C and D will meet for the first time at the same moment.

   Criteria. 46 minutes are used in calculations instead of 53 — 3 points. If it is stated that any of cars' speeds are equal — 1 point. Only the answer is given without any explanation — 0 points.

7. A school was opened on the island of knights and liars (a knight always tells the truth, a liar always lies). All $2N$ students are of different heights. They lined up in pairs one after another (in other words, in two equal columns). The two people standing first said: "I am taller than 2 people: my neighbor in a pair and the person behind me". The last two said: "I am also taller than 2 people: my neighbor in a pair and the person in front of me". Finally, everyone else said: "I am taller than 3 people: my neighbor in a pair, the person in front of me and the person behind me".
   a) Find the largest possible number of knights among the students.
   b) Is it possible for all the students to be liars?

Solution. A) There is no more than one knight in each pair, so there are no more than N knights (this amount can be achieved by placing N higher students in chess order).
B) No. All the students are of the different heights, so the heighest of them is always a knight.

Criteria. Problem a) is fully solved — 5 points. If only one part of the solution is given (either an estimation or an example) — 2 points. One or few examples are given as a proof — 0 points. Problem b) is fully solved — 2 points.

8. Kate wrote a number divisible by 30 on the board and encrypted it according to the rules of alphametic puzzles (different letters correspond to different digits, the same letters — the same digits). She got the word "GUATEMALA". How many different numbers could Kate write on the board?

Solution. The letter A should be equal 0. The other 6 letters are non-null digits whose sum is divisible by 3. Note that each remainder modulo 3 occurs 3 times. Checking all possible sets of remainders we conclude that the combinations whose sum is divisible by 3 are 000111, 000222, 111222, 001122. Now count 6-element subsets of $\{1, 2, \ldots, 9\}$ for each combination: there is only 1 set for each of the first three combinations and $3^3 = 27$ sets for the last one. Each subset of digits can be rearranged in 6! ways, so the answer is $30 \cdot 6! = 21600$.

Criteria. 1 point if it is shown that $A = 0$. If combinations of the remaining letters are found by brute force — it is OK, but if some cases are lost — no more than 3 points. -1 point in case of calculation error(s) or if the answer is an expression which is not counted.

Problems for grade R9

Each task is assessed at 7 points. Some problems have their own criteria (printed in gray).

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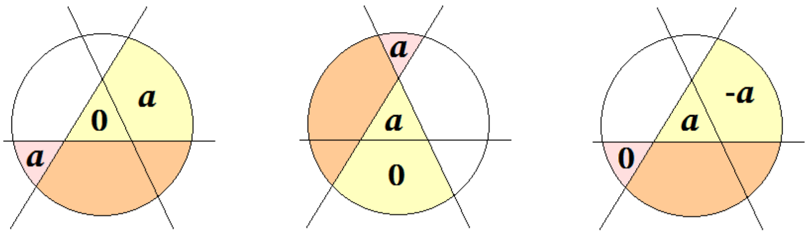
1. Is there a year in the 21st century whose number can be represented as $\dfrac{a+b\cdot c\cdot d\cdot e}{f+g\cdot h\cdot i\cdot j}$ where $a, b, c, d, e, f, g, h, i, j$ are the digits 0 to 9 in any order?

   Solution. Yes, for example, $2022 = \dfrac{6+4\cdot 7\cdot 8\cdot 9}{1+0\cdot 2\cdot 3\cdot 5}$. There are analogous examples for 2018, 2020, 2021.

2. A circle is divided into 7 parts by 3 lines. Maria wrote 7 different integers into these parts (one number in each part) so that the sum of numbers on one side of each line is equal to the sum of numbers on the other side. One of the numbers is 0. Prove that some other number is negative.

   Solution. Let's analyze three cases of zero location. On the pictures, the sum of the yellow sectors is equal to the sum of the pink ones (because yellow + orange = pink + orange = half the sum of all numbers). We see that the first two cases are impossible (because some numbers are equal), and in the third case one of the numbers a and $-a$ is negative.



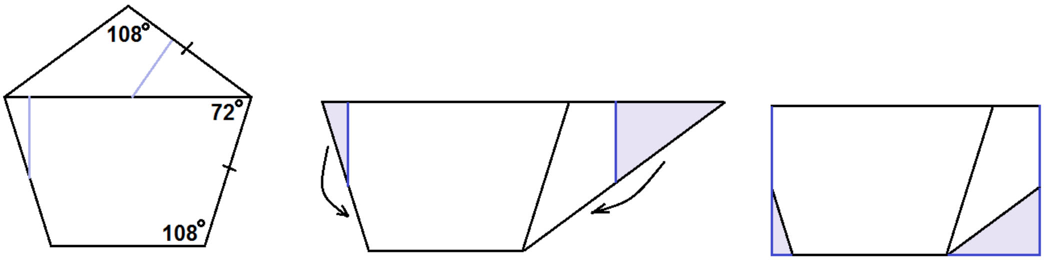
   Criteria. Not more than 2 points if not all the cases are considered.

3. A chess championship is held in a village club: each participant must play one game with each other. There is only one board in the club, so two games cannot be played at the same time. According to the rules of the championship, at any moment the number of games already played by different participants must differ by no more than 1. First several games of the championship were played in accordance with the rules. Is it always possible to complete the championship, following the rules?

   Solution. No, not always. For example, let there be 6 players in the championship, and the first games were played in the following order: 12, 34, 56, 13, 24. Now the least number of games were played by 5 and 6, so we should have a match between them, but it has already taken place.

4. Prove that it is possible to cut a regular pentagon into 4 parts and rearrange them to make a rectangle without gaps and overlays.

   Solution. By cutting off and shifting the triangle we obtain a trapezoid (this follows from the angles marked on the picture and the equality of the sides). Next, cutting off two right triangles from the trapezoid, we rotate them and get a rectangle (see figure).



5. Four cars A, B, C and D start simultaneously from the same point of a circular track. A and B travel clockwise, while C and D — counter-clockwise. All cars move at constant (but pairwise different) speeds. After exactly 7 minutes of the race A meets C for the first time, and at the same moment B meets D for the first time. 46 minutes later, A and B meet for the first time. How long does it take from the start to the first meeting of C and D?

   Solution. If A meets C and B meets D every 7 minutes, then their convergence speeds are equal: $V_A + V_C = V_B + V_D$. Therefore, the removal speeds of A with B and C with D are equal: $V_A - V_B = V_D - V_C$. So, since A meets B for the first time at the 53^{rd} minute, then C and D will meet for the first time at the same moment.

   Criteria. 46 minutes are used in calculations instead of 53 — 3 points. If it is stated that any of cars' speeds are equal — 1 point. Only the answer is given without any explanation — 0 points.

6. How many solutions in positive integers the equation $(a+1)(b+1)(c+1) = 2abc$ has?

   Solution. Let's rewrite the equation as $(1+1/a)(1+1/b)(1+1/c) = 2$. By symmetry, it suffices to find all solutions with $a \leqslant b \leqslant c$. Then $(1+1/a)^3 \geqslant 2$, i.e. $a \leqslant (\sqrt[3]{2}-1)^{-1} < 4$ and $a \in \{1, 2, 3\}$. In the case $a = 1$ the inequality $2(1+1/b)^2 \geqslant 2$ is satisfied, i.e. there are no solutions. If $a = 2$ then $\frac{3}{2}(1+1/b)^2 \geqslant 2$, i.e. $2 \leqslant b \leqslant (\frac{2\sqrt{3}}{3}-1)^{-1} < 7$. In this case, there are 3 solutions $(a, b, c) = (2, 4, 15), (2, 5, 9), (2, 6, 7)$ (for $b = 2$ and $b = 3$ the equation on c has no solutions in natural numbers). Finally, if $a = 3$, then $\frac{4}{3}(1+1/b)^2 \geqslant 2$, i.e. $3 \leqslant b \leqslant (\sqrt{\frac{3}{2}}-1)^{-1} < 5$. This gives 2 more solutions $(a, b, c) = (3, 3, 8), (3, 4, 5)$. Including permutations, there are 27 solutions in total.

   Criteria. Not more than 2 points if only a part of the solutions is found.

7. Let us call a positive integer *useful* if its decimal notation contains neither zeroes nor equal digits, and if the product of all its digits is divisible by the sum of these digits. Find two maximal consecutive (i. e. differing by 1) useful numbers.

   Answer: 9875213 and 9875214.

   Solution. These two numbers satisfy all conditions. Let us prove that they are maximal. For consecutive numbers, the sums of digits are consecutive (otherwise we have a transition through the digit, that is, 0 at the end). But then the maximum possible sums of digits are 35 and 36 (among any two larger ones, from 37 to $45 = 1 + \ldots + 9$, at least one has a prime divisor greater than 9). Consecutive useful numbers have no more than seven digits, otherwise the sum of their digits will be at least $1 + \ldots + 8 = 36$ each. There are no consecutive useful numbers of the form $9876***$, since the sum of digits of such numbers is at least $9+8+7+6+1+2+3 = 36$. And consecutive useful numbers of the form $9875***$ can differ from those found only by permutation of the last digits (otherwise the sum of their digits is greater than 36), so the numbers we found are really the largest.

   Criteria. 2 points for the answer and 5 points for an estimation.

8. A park has a shape of a 10×10 cells square. A street light can be placed in any cell (but no more than one light in each cell).

a) A park is called *illuminated* if, no matter in which cell a visitor stands, there exists a square of 9 cells containing the visitor and a light. What minimal number of lights is required to illuminate the park?

b) A park is called *securely illuminated* if it remains illuminated even when one arbitrary street light is broken. What is the minimal number of lights in a securely illuminated park?

Solution. a) 4. Let's divide the park into 4 quarters (5×5 squares), then each quarter contains at least one light (otherwise e. g. the corner is not illuminated). Placing a light in the center of each quarter, we get an example.

b) 10.

Estimation. Each 3×3 corner square must contain at least two lights (to illuminate the corner cell). Let us temporarily leave only these 8 lights. Each of them illuminates only within its quarter, and if a light in the center of a quarter is broken (or if it doesn't exist), then there is a "dark" five-cell strip along the border of this quarter. Note that the union of two such strips for opposite quarters in any case cannot be illuminated with one light, so we need at least two more lights.

An example:

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Criteria. Part (a) costs 2 points (1 for the example an 1 for an estimation). In the part (b), 2 points are given for an example and 3 points for an estimation (1 point of them for a that 8 lights are not enough).

International Mathematical Olympiad
«Formula of Unity» / «The Third Millennium»
Year 2022/2023. Qualifying round

Problems for grade R10

Each task is assessed at 7 points. Some problems have their own criteria (printed in gray).

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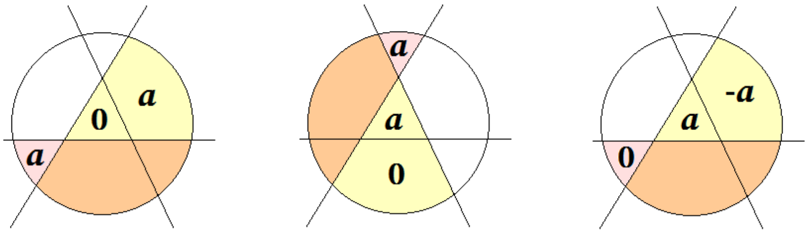
1. A circle is divided into 7 parts by 3 lines. Maria wrote 7 different integers into these parts (one number in each part) so that the sum of numbers on one side of each line is equal to the sum of numbers on the other side. One of the numbers is 0. Prove that some other number is negative.

   Solution. Let's analyze three cases of zero location. On the pictures, the sum of the yellow sectors is equal to the sum of the pink ones (because yellow + orange = pink + orange = half the sum of all numbers). We see that the first two cases are impossible (because some numbers are equal), and in the third case one of the numbers a and $-a$ is negative.



   Criteria. Not more than 2 points if not all the cases are considered.

2. A chess championship is held in a village club: each participant must play one game with each other. There is only one board in the club, so two games cannot be played at the same time. According to the rules of the championship, at any moment the number of games already played by different participants must differ by no more than 1. Prove that, for any number of participants, it is possible to hold the championship in compliance with the rules.

   Solution. a) Consider the case with an even number of players n. Let's divide the championship into separate circles, so that in circle number i there are participants whose sum of numbers equals i modulo n. Then in each round each player plays exactly once, and at the end of the round everyone has played the same number of games.

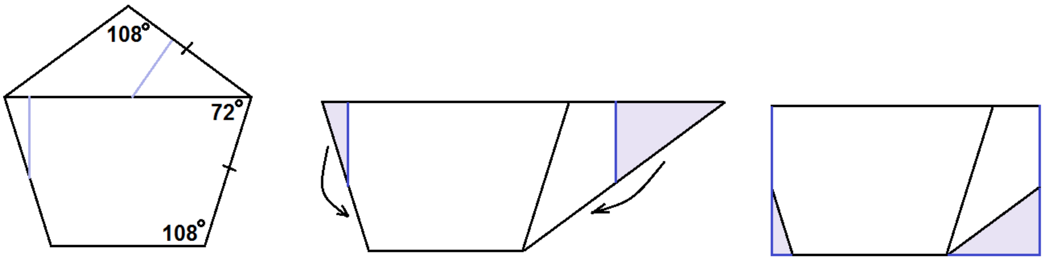
   b) Let there be an odd number of players.
   1. Let's play all matches with the sum of numbers 2 modulo n (player 1 rests). Let's play a $1n$ match. Let's play all the matches with the sum of the numbers 0 modulo n (player n rests). As a result of this "double round", everyone played two matches, and all the rules are followed.

   2. Let's do the same, replacing the players 1 and n with 2 and $n-1$, and the sums — with $2 \cdot 2$ and $2(n-1)$.
   We continue to do so. As a result, in the kth round ($k = 1, 2, \ldots, (n-1)/2$), all matches with the sum of remainders $2k$ and $2(n+1-k)$ take place, as well as the next match with the sum of remainders 1 (which cannot be obtained in this way).

   Criteria. The case of even n is worth 3 points, the case of odd n — 4 points. If only the idea of splitting participants into pairs is given, then 1 point is given.

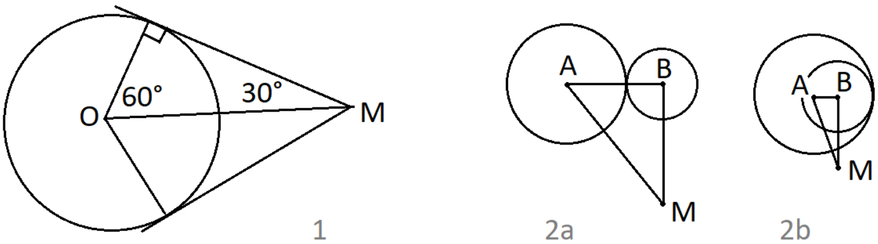
3. Prove that it is possible to cut a regular pentagon into 4 parts and rearrange them to make a rectangle without gaps and overlays.

Solution. By cutting off and shifting the triangle we obtain a trapezoid (this follows from the angles marked on the picture and the equality of the sides). Next, cutting off two right triangles from the trapezoid, we rotate them and get a rectangle (see figure).



4. We will call a point *convenient* for a circle if the angle between the tangents drawn from this point to the circle is equal to 60°. Two circles with centers A and B are tangent, and the point M is convenient for each of them. Find the ratio of the radii of the circles if $\triangle ABM$ is a right triangle.

   Solution. Note (see fig. 1) that point M is *convenient* for a circle with center O if and only if OM is twice the radius.



1 2a 2b

Without loss of generality, let the circles have radii $r_A \geqslant r_B$. Since they are touching, $AB = r_A \pm r_B$ (plus when touching externally and minus when touching internally, see fig. 2). On the other hand, $AM = 2r_A$ and $BM = 2r_B$ because M is convenient for both circles. In triangle AMB, side AM is the hypotenuse, since $2r_A \geqslant 2r_B$ and $2r_A \geqslant r_A \pm r_B$. So we get the equation

$$4r_A^2 = 4r_B^2 + r_A^2 \pm 2r_Ar_B + r_B^2,$$

i.e. $3r_A^2 = 5r_B^2 \pm 2r_Ar_B$. If the tangency is internal, then $r_A = r_B$, but then $A = B$ and AMB is not a triangle. If the tangency is external, then $r_A = \frac{5}{3}r_B$. Therefore, the ratio of the radii is $3:5$.

Criteria. If it is not taken into account that the circles can touch internally, then 2 points are deducted.

5. How many solutions in positive integers the equation $(a+1)(b+1)(c+1) = 2abc$ has?

   Solution. Let's rewrite the equation as $(1+1/a)(1+1/b)(1+1/c) = 2$. By symmetry, it suffices to find all solutions with $a \leqslant b \leqslant c$. Then $(1+1/a)^3 \geqslant 2$, i.e. $a \leqslant (\sqrt[3]{2}-1)^{-1} < 4$ and $a \in \{1,2,3\}$. In the case $a = 1$ the inequality $2(1+1/b)^2 \geqslant 2$ is satisfied, i.e. there are no solutions. If $a = 2$ then $\frac{3}{2}(1+1/b)^2 \geqslant 2$, i.e. $2 \leqslant b \leqslant (\frac{2\sqrt{3}}{3}-1)^{-1} < 7$. In this case, there are 3 solutions $(a,b,c) = (2,4,15), (2,5,9), (2,6,7)$ (for $b = 2$ and $b = 3$ the equation on c has no solutions in natural numbers). Finally, if $a = 3$, then $\frac{4}{3}(1+1/b)^2 \geqslant 2$, i.e. $3 \leqslant b \leqslant (\sqrt{\frac{3}{2}}-1)^{-1} < 5$. This gives 2 more solutions $(a,b,c) = (3,3,8), (3,4,5)$. Including permutations, there are 27 solutions in total.

   Criteria. Not more than 2 points if only a part of the solutions is found.

6. A park has a shape of a 10×10 cells square. A street light can be placed in any cell (but no more than one light in each cell).

a) A park is called *illuminated* if, no matter in which cell a visitor stands, there exists a square of 9 cells containing the visitor and a light. What minimal number of lights is required to illuminate the park?
b) A park is called *securely illuminated* if it remains illuminated even when one arbitrary street light is broken. What is the minimal number of lights in a securely illuminated park?

Solution. a) 4. Let's divide the park into 4 quarters (5×5 squares), then each quarter contains at least one light (otherwise e. g. the corner is not illuminated). Placing a light in the center of each quarter, we get an example.

b) 10.
Estimation. Each 3×3 corner square must contain at least two lights (to illuminate the corner cell). Let us temporarily leave only these 8 lights. Each of them illuminates only within its quarter, and if a light in the center of a quarter is broken (or if it doesn't exist), then there is a "dark" five-cell strip along the border of this quarter. Note that the union of two such strips for opposite quarters in any case cannot be illuminated with one light, so we need at least two more lights.

An example:

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Criteria. Part (a) costs 2 points (1 for the example an 1 for an estimation). In the part (b), 2 points are given for an example and 3 points for an estimation (1 point of them for a that 8 lights are not enough).

7. $f(x)$ is a linear function such that the equation $f(f(x)) = x + 1$ has no solutions. Find all possible values of $f(f(f(f(f(2022))))) - f(f(f(2022))) - f(f(2022))$.

Solution. Let $f(x) = kx + b$, then $f(f(x)) = k(kx+b)+b = k^2x+kb+b$. The equation may not have solutions only for $k^2 = 1$, that is, for the functions $x + b$ or $-x + b$, so the answer is either $(2022+5b)-(2022+3b)-(2022+2b) = -2022$, or $(-2022+b)-(-2022+b)-2022 = -2022$.

Answer: -2022.

Criteria. No more than 2 points are taken away for gaps in a proof that $k = \pm 1$.

8. Let's call *efficiency* of a positive integer n the fraction of all integers from 1 to n that have a common divisor greater than 1 with n. For example, the efficiency of the number 6 is $\frac{2}{3}$.
a) Is there a number with efficiency more than 80%? If so, find the smallest such number.
b) Is there a number whose efficiency is maximal (that is, not less than that of any other number)? If so, find the smallest such number.

Solution. Let's study **in**efficiency (i. e. 1 minus efficiency). It follows from the formula of Euler's function that inefficiency equals to $\frac{p_1-1}{p_1} \cdot \ldots \frac{p_k-1}{p_k}$, where $p_1, \ldots, p_k$ are all possible different prime divisors of n. Then, by adding a new prime factor, we can increase efficiency, that is, in point (b) the answer is no.

The smallest number with efficiency greater than 80% is $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30030$. Its efficiency is $1 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{12}{13}$. Let us prove that it is more efficient than all its smaller numbers.

Indeed, if a number is divisible by p^k $(k > 1)$ then we can divide it by k without changing the efficiency. So all prime factors of the desired number are different. Further, if the factors are not consecutive prime numbers, then replacing one of them with a smaller one will increase efficiency. This means that "efficiency records" can only be set by numbers of the form "product of the first few primes", and the efficiency of $2 \cdot 3 \cdot 5 \cdot 7$ is less than 80%.

Criteria. 5 points for part (a), 2 points for (b).

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«Formula of Unity» / «The Third Millennium»
Year 2022/2023. Qualifying round

Problems for grade R11

Each task is assessed at 7 points. Some problems have their own criteria (printed in gray).

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1. Let us call a positive integer *useful* if its decimal notation contains neither zeroes nor equal digits, and if the product of all its digits is divisible by the sum of these digits. Find two maximal consecutive (i. e. differing by 1) useful numbers.

   Answer: 9875213 and 9875214.

   Solution. These two numbers satisfy all conditions. Let us prove that they are maximal. For consecutive numbers, the sums of digits are consecutive (otherwise we have a transition through the digit, that is, 0 at the end). But then the maximum possible sums of digits are 35 and 36 (among any two larger ones, from 37 to $45 = 1 + \ldots + 9$, at least one has a prime divisor greater than 9). Consecutive useful numbers have no more than seven digits, otherwise the sum of their digits will be at least $1 + \ldots + 8 = 36$ each. There are no consecutive useful numbers of the form $9876{*}{*}{*}$, since the sum of digits of such numbers is at least $9+8+7+6+1+2+3 = 36$. And consecutive useful numbers of the form $9875 * \ * *$ can differ from those found only by permutation of the last digits (otherwise the sum of their digits is greater than 36), so the numbers we found are really the largest.

   Criteria. 2 points for the answer and 5 points for an estimation.

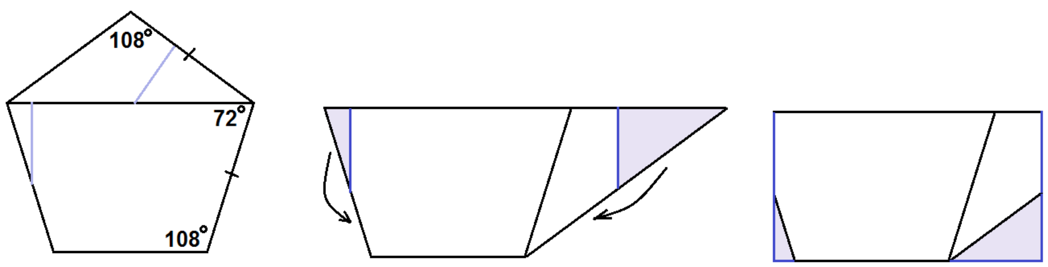
2. Four cars A, B, C and D start simultaneously from the same point of a circular track. A and B travel clockwise, while C and D — counter-clockwise. All cars move at constant (but pairwise different) speeds. After exactly 7 minutes of the race A meets C for the first time, and at the same moment B meets D for the first time. 46 minutes later, A and B meet for the first time. How long does it take from the start to the first meeting of all four cars?

   Solution. A and C meet once every 7 minutes, and A and B — once every 53 minutes. So, all together they will meet at such time that is a multiple of both 7 and 53, hence once in $7 \cdot 53 = 371$ minutes. From the other hand, B meets D every 7 minutes as well, so at the 371$^{\text{st}}$ minutes D will be at the same place as the rest 3 cars.

   Criteria. 46 minutes are used in calculations instead of 53 — 3 points. If it is stated that any of cars' speeds are equal — 1 point. Only the answer is given without any explanation — 0 points.

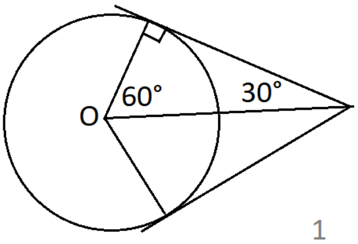
3. Prove that it is possible to cut a regular pentagon into 4 parts and rearrange them to make a rectangle without gaps and overlays.

   Solution. By cutting off and shifting the triangle we obtain a trapezoid (this follows from the angles marked on the picture and the equality of the sides). Next, cutting off two right triangles from the trapezoid, we rotate them and get a rectangle (see figure).



4. We will call a point *convenient* for a circle if the angle between the tangents drawn from this point to the circle is equal to 60°. Two circles with centers A and B are tangent, and the point M is convenient for each of them. Find the ratio of the radii of the circles if $\triangle ABM$ is a right triangle.

Solution. Note (see fig. 1) that point M is *convenient* for a circle with center O if and only if OM is twice the radius.



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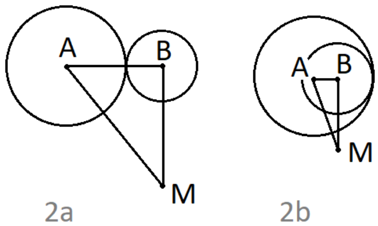

2a 2b

Without loss of generality, let the circles have radii $r_A \geqslant r_B$. Since they are touching, $AB = r_A \pm r_B$ (plus when touching externally and minus when touching internally, see fig. 2). On the other hand, $AM = 2r_A$ and $BM = 2r_B$ because M is convenient for both circles. In triangle AMB, side AM is the hypotenuse, since $2r_A \geqslant 2r_B$ and $2r_A \geqslant r_A \pm r_B$. So we get the equation

$$4r_A^2 = 4r_B^2 + r_A^2 \pm 2r_Ar_B + r_B^2,$$

i.e. $3r_A^2 = 5r_B^2 \pm 2r_Ar_B$. If the tangency is internal, then $r_A = r_B$, but then $A = B$ and AMB is not a triangle. If the tangency is external, then $r_A = \frac{5}{3}r_B$. Therefore, the ratio of the radii is $3:5$.

Criteria. If it is not taken into account that the circles can touch internally, then 2 points are deducted.

5. Find all real a, b, c such that

$$27^{a^2+b+c+1} + 27^{b^2+c+a+1} + 27^{c^2+a+b+1} = 3.$$

Solution. According the means inequality,

$$\frac{27^{a^2+b+c+1} + 27^{b^2+c+a+1} + 27^{c^2+a+b+1}}{3} \geqslant \left(27^{a^2+b+c+1} \cdot 27^{b^2+c+a+1} \cdot 27^{c^2+a+b+1}\right)^{1/3} =$$

$$= 3^{a^2+b+c+1+b^2+a+c+1+c^2+a+b+1} = 3^{(a+1)^2+(b+1)^2+(c+1)^2} \geqslant 1,$$

and the inequality turns into equality only if $a = b = c = -1$.

6. A park has a shape of a 10×10 cells square. A street light can be placed in any cell (but no more than one light in each cell).
a) A park is called *illuminated* if, no matter in which cell a visitor stands, there exists a square of 9 cells containing the visitor and a light. What minimal number of lights is required to illuminate the park?
b) A park is called *securely illuminated* if it remains illuminated even when one arbitrary street light is broken. What is the minimal number of lights in a securely illuminated park?

Solution. a) 4. Let's divide the park into 4 quarters (5×5 squares), then each quarter contains at least one light (otherwise e. g. the corner is not illuminated). Placing a light in the center of each quarter, we get an example.

b) 10.
Estimation. Each 3×3 corner square must contain at least two lights (to illuminate the corner

cell). Let us temporarily leave only these 8 lights. Each of them illuminates only within its quarter, and if a light in the center of a quarter is broken (or if it doesn't exist), then there is a "dark" five-cell strip along the border of this quarter. Note that the union of two such strips for opposite quarters in any case cannot be illuminated with one light, so we need at least two more lights.

An example:

		*					*		
		*					*		
		*					*		
		*					*		
		*					*		

Criteria. Part (a) costs 2 points (1 for the example an 1 for an estimation). In the part (b), 2 points are given for an example and 3 points for an estimation (1 point of them for a that 8 lights are not enough).

7. Let's call *efficiency* of a positive integer n the fraction of all integers from 1 to n that have a common divisor greater than 1 with n. For example, the efficiency of the number 6 is $\frac{2}{3}$.
a) Is there a number with efficiency more than 80%? If so, find the smallest such number.
b) Is there a number whose efficiency is maximal (that is, not less than that of any other number)? If so, find the smallest such number.

Solution. Let's study **in**efficiency (i. e. 1 minus efficiency). It follows from the formula of Euler's function that inefficiency equals to $\frac{p_1-1}{p_1}\cdot\ldots\frac{p_k-1}{p_k}$, where $p_1,\ldots,p_k$ are all possible different prime divisors of n. Then, by adding a new prime factor, we can increase efficiency, that is, in point (b) the answer is no.

The smallest number with efficiency greater than 80% is $2\cdot3\cdot5\cdot7\cdot11\cdot13=30030$. Its efficiency is $1-\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{10}{11}\cdot\frac{12}{13}$. Let us prove that it is more efficient than all its smaller numbers. Indeed, if a number is divisible by p^k $(k>1)$ then we can divide it by k without changing the efficiency. So all prime factors of the desired number are different. Further, if the factors are not consecutive prime numbers, then replacing one of them with a smaller one will increase efficiency. This means that "efficiency records" can only be set by numbers of the form "product of the first few primes", and the efficiency of $2\cdot3\cdot5\cdot7$ is less than 80%.

Criteria. 5 points for part (a), 2 points for (b).

8. There is a continuous function f such that $f(f(f(f(f(0)))))=0$. Prove that the equation $f(f(x))=x$ has at least one solution.

Solution. Let us prove that $f(x)=x$ has a solution. Indeed, if it is not, then $f(x)$ is either always greater or always less than x (because $f(x)-x$ is continuous). So, when f is applied, the result always changes in the same direction. But in this case the condition of the problem cannot be true. So there exists x_0: $f(x_0)=x_0$. But then $f(f(x_0))=x_0$.